Computer Science

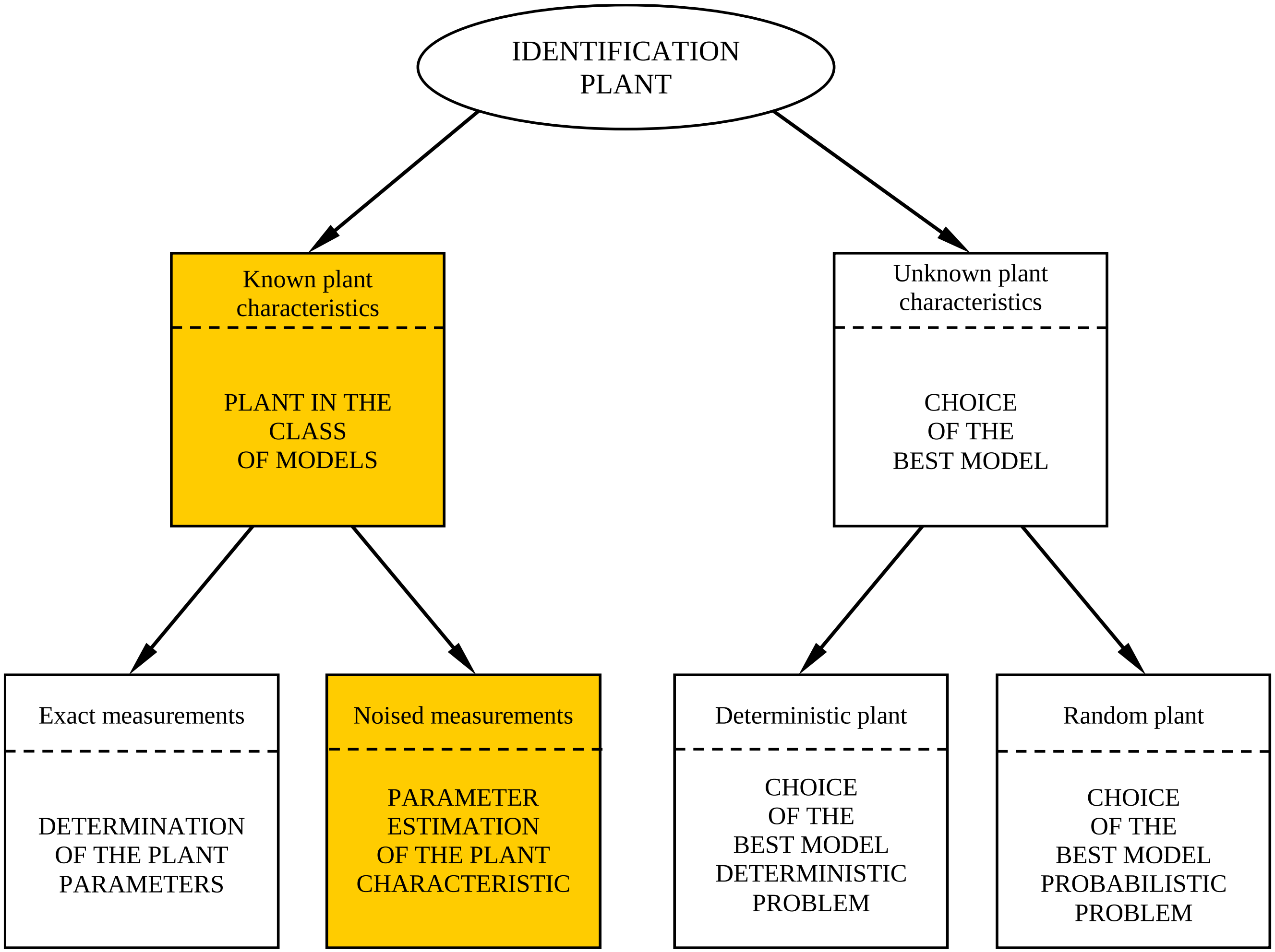
Jerzy Świątek

Systems Modelling and Analysis

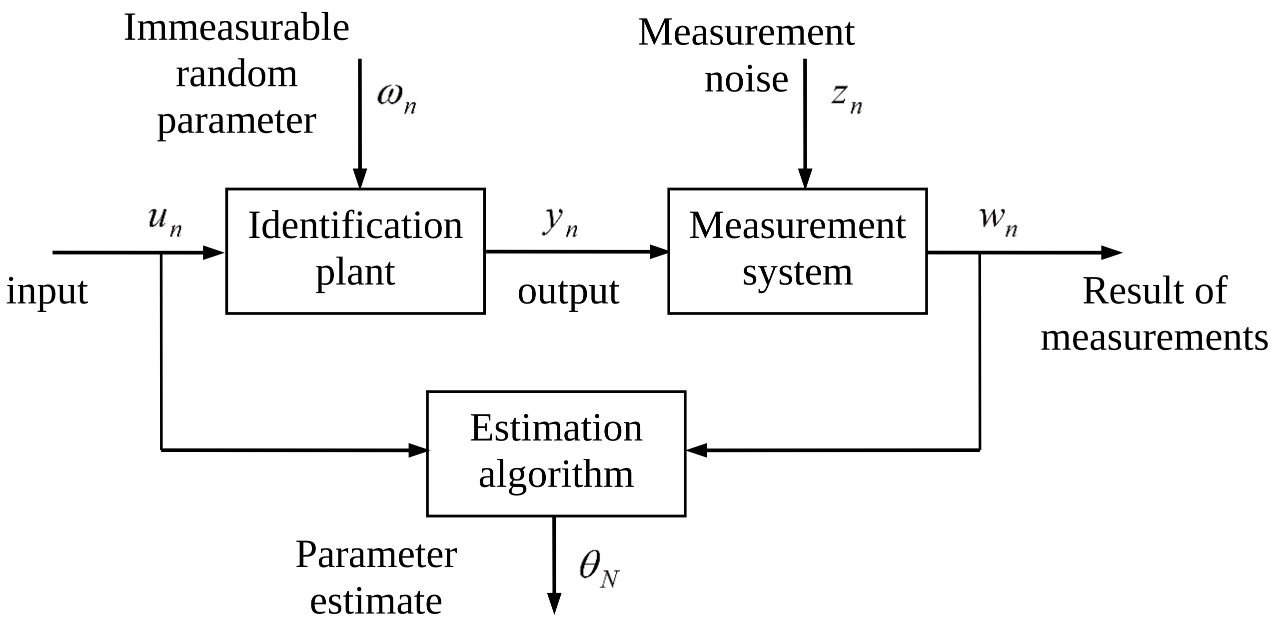
Choose yourself and new technologies

L.9. Estimation of the plant parameter with random value

HUMAN CAPITAL
HUMAN – BEST INVESTMENT!
Wrocław University of Technology
EUROPEAN SOCIAL FUND
Project co-financed from the EU European Social Fund

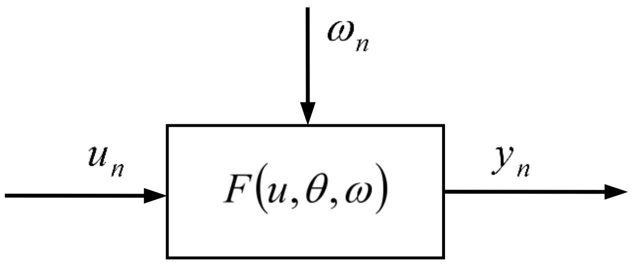

IDENTIFICATION PLANT
Known plant characteristics
PLANT IN THE CLASS OF MODELS
Unknown plant characteristics
CHOICE OF THE BEST MODEL
Exact measurements
DETERMINATION OF THE PLANT PARAMETERS
Noised measurements
PARAMETER ESTIMATION OF THE PLANT CHARACTERISTIC
Deterministic plant
CHOICE OF THE BEST MODEL DETERMINISTIC PROBLEM
Random plant
CHOICE OF THE BEST MODEL PROBABILISTIC PROBLEM

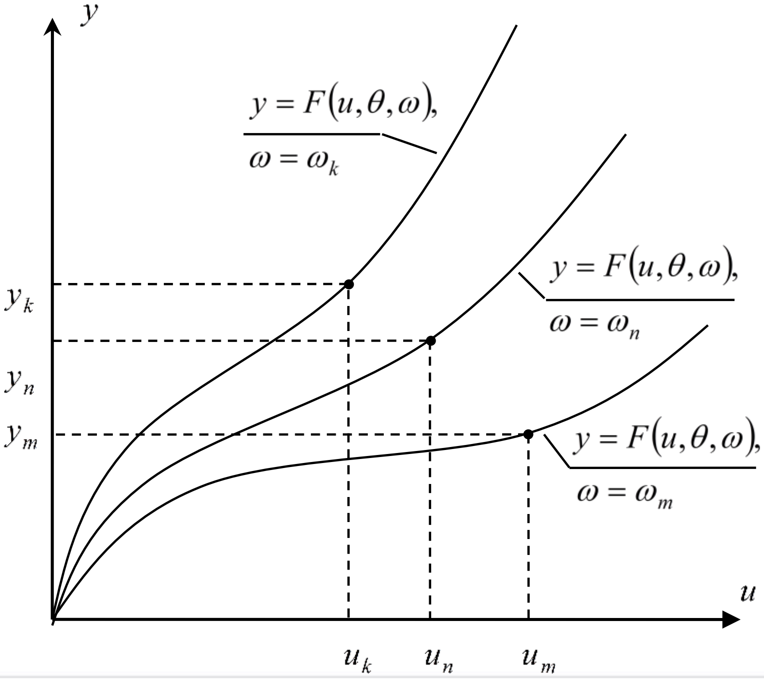
Plant parameter estimation problem



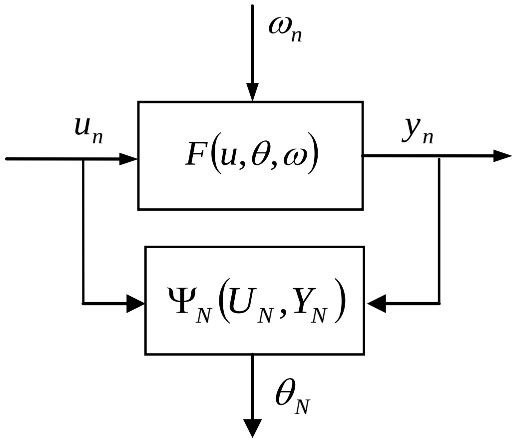
Immeasurable random plant parameter

- Measurements of plant characteristic with random parameter





Immeasurable random plant parameter



- Problem formulation

Plant characteristic: $y = F(u,\theta,\omega)$

Random plant parameter: $\omega \in \Omega \subseteq R^L$, $(\dim y = \dim \omega = L)$

F – one-to-one mapping $\omega = F_\omega^{-1}(u,\theta,y)$

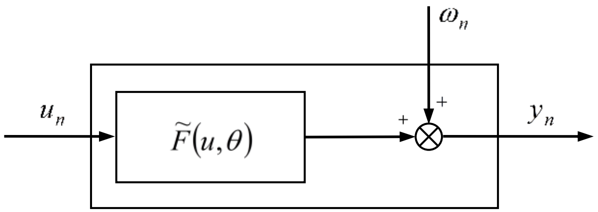
ω_n – value of random variable $\underline{\omega}$ from $\Omega \subseteq R^L$

Probability density function $f_\omega(\omega)$ is given

Measurements: $U_N = [u_1 \quad u_2 \quad \cdots \quad u_N]$, $Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$

Estimation algorithm: $\theta_N = \Psi_N(U_N, Y_N)$

Least square method



Assumptions:

$$y = F(u,\theta,\omega) = \tilde{F}(u,\theta) + \omega$$ – plant characteristic

$$\underset{\underline{\omega}}{E}[\underline{\omega}] = 0 \qquad \underset{\underline{\omega}}{Var}[\underline{\omega}] < \infty$$

Calculations:

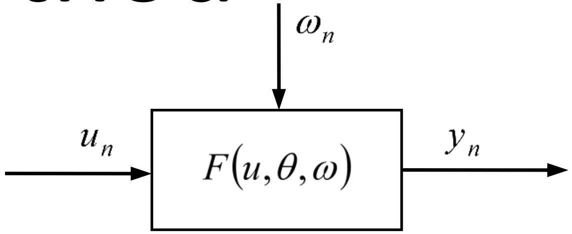
Least square method minimizes empirical variance:

$$Var_{\omega N}(U_N, Y_N, \theta) = \frac{1}{N}\sum_{n=1}^{N}\left(y_n - \tilde{F}(u_n,\theta)\right)^2$$

Estimation algorithm has the form:

$$\theta_N = \Psi_N(U_N, Y_N) \quad \rightarrow \quad Var_{\omega N}(U_N, Y_N, \theta_N) = \min_{\theta\in\Theta} Var_{\omega N}(U_N, Y_N, \theta)$$

Maximum likelihood method



<u>Assumptions:</u>

ω_n is value of random variable $\underline{\omega}$, with probability density function $f_\omega(\omega)$

For a given input u_n, output $y_n, n=1,2,\ldots,N$ is measured

Sequence $y_n, n=1,2,\ldots,N$ contains values of random variable $\underline{y}$: $\underline{y}=F(u,\theta,\underline{\omega})$

<u>Calculations:</u>

Probability density function $f_y(y,\theta;u)= f_\omega\left(F_\omega^{-1}(u,\theta,y)\right)\cdot\left|J_F\right|$

where J_F is Jacobi matrix: $J_F=\dfrac{\partial F_\omega^{-1}(u,\theta,y)}{\partial y}$

Likelihood function: $L_N(Y_N,\theta;U_N)=\prod_{n=1}^{N} f_y(y_n,\theta;u_n)=\prod_{n=1}^{N} f_\omega\left(F_\omega^{-1}(u_n,\theta,y_n)\right)\left|J_F\right|$

Estimation algorithm: $\theta_N=\Psi_N(U_N,Y_N) \;\rightarrow\; L_N(Y_N,\theta_N;U_N)=\max_{\theta\in\Theta} L_N(Y_N,\theta;U_N)$

Bayesian method

Assumptions:

Additionally θ is value of random variable $\underline{\theta}$, with probability density function $f_\theta(\theta)$

Calculations:

$$r(\overline{\theta},Y_N;U_N)\overset{df}{=}\underset{\underline{\theta}}{E}\left[L(\underline{\theta},\overline{\theta})\middle|Y_N;U_N\right]=\int_\Theta L\left(\theta,\overline{\theta}=\overline{\Psi}(U_N,Y_N)\right)f'(\theta|Y_N;U_N)\,d\theta$$

$$f_y(y|\theta;u)=f_\omega\left(F_\omega^{-1}(u,\theta,y)\right)|J_F|$$

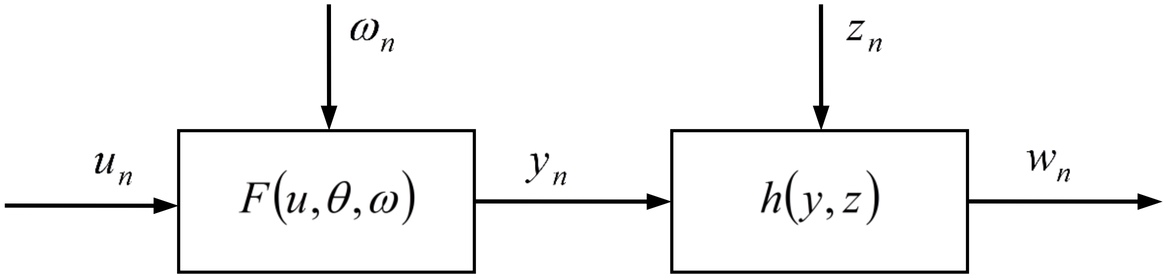
A'posteriori probability density function:

$$f'(\theta|Y_N;U_N)=\frac{f_\theta(\theta)\cdot\prod_{n=1}^{N}f_\omega\left(F_\omega^{-1}(u_n,\theta,y_n)\right)|J_F|}{\int_\Theta f_\theta(\theta)\cdot\prod_{n=1}^{N}f_\omega\left(F_\omega^{-1}(u_n,\theta,y_n)\right)|J_F|\,d\theta}$$

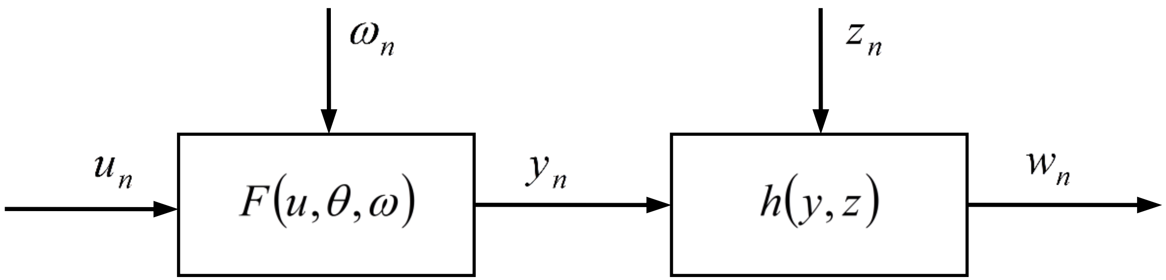
Estimation algorithm: $\theta_N=\Psi_N(U_N,Y_N)\quad\rightarrow\quad r(\theta_N,Y_N;U_N)=\min_{\overline{\theta}\in\Theta}r(\overline{\theta},Y_N;U_N)$

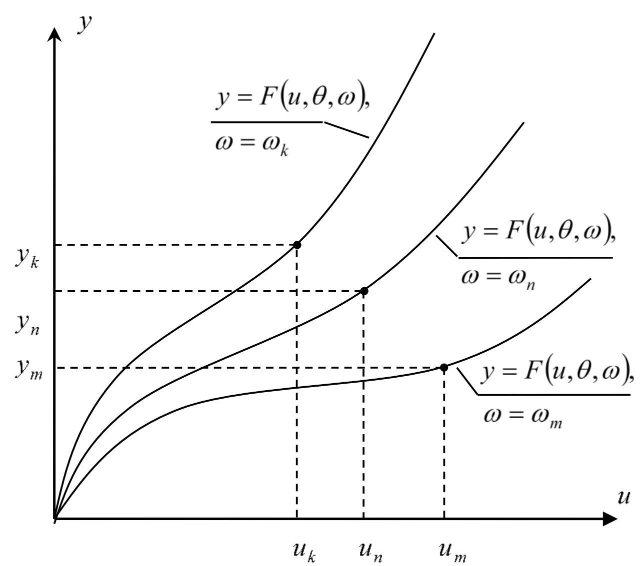
Random plant parameter and measurement noise

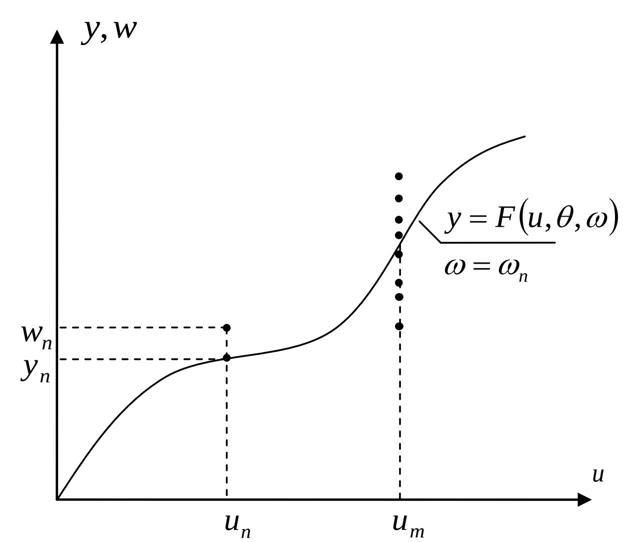
- Noised measurement plant output with randomly changed parameters



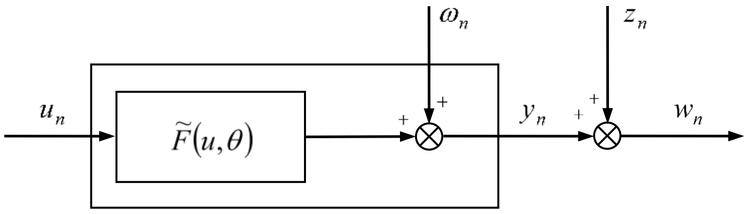
- Noised measurement plant output with randomly changed parameters







Least square method



<u>Assumptions:</u>

$\underline{z}$, $\underline{\omega}$ – random variables such that $\underset{\underline{\omega}}{E}[\underline{\omega}]=0,\ \underset{\underline{\omega}}{Var}[\underline{\omega}]<\infty,\ \underset{\underline{z}}{E}[\underline{z}]=0,\ \underset{\underline{z}}{Var}[\underline{z}]<\infty$

Measurements are values of random variable: $\underline{w}=\tilde{F}(u,\theta)+\underline{\omega}+\underline{z}$

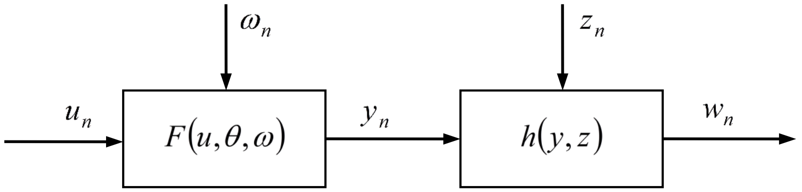
$$\underset{\underline{\omega}+\underline{z}}{E}[\underline{\omega}+\underline{z}]=0,\ \underset{\underline{\omega}+\underline{z}}{Var}[\underline{\omega}+\underline{z}]<\infty$$

<u>Calculations:</u>

Empirical variance: $Var_{(\omega+z)N}(U_N,W_N,\theta)=\frac{1}{N}\sum_{n=1}^{N}\left(w_n-\tilde{F}(u_n,\theta)\right)^2$

Estimation algorithm: $\theta_N=\Psi_N(U_N,Y_N)\ \rightarrow\ Var_{(\omega+z)N}(U_N,W_N,\theta_N)=\min_{\theta\in\Theta}Var_{(\omega+z)N}(U_N,W_N,\theta)$

Maximum likelihood method



Assumptions:

ω_n is value of random variable $\underline{\omega}$, with probability density function $f_\omega(\omega)$

z_n is value of random variable $\underline{z}$, with probability density function $f_z(z)$

$w_n, n=1, 2, \ldots, N$ are values of random variable $\underline{w}=h(\underline{y},\underline{z})$

For a given u_n, $y_n, n=1, 2, \ldots, N$ are values of random variable $\underline{y}=F(u,\theta,\underline{\omega})$

Calculations:

Function h is no one to one mapping with respect $(\underline{y},\underline{z})$, so we add identity $\underline{y}=\underline{y}$ and we have:

$$\underline{w}=h(\underline{y},\underline{z})$$
$$\underline{y}=\underline{y}$$

The inverse transformation:

$$\underline{z}=h_z^{-1}(\underline{y},\underline{w})$$
$$\underline{y}=\underline{y}$$

Maximum likelihood method

Jacobi matrix:

$$J=\begin{bmatrix}\frac{\partial h_z^{-1}(y,w)}{\partial w} & \frac{\partial h_z^{-1}(y,w)}{\partial y}\\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial y}\end{bmatrix}=\begin{bmatrix}\frac{\partial h_z^{-1}(y,w)}{\partial w} & \frac{\partial h_z^{-1}(y,w)}{\partial y}\\ O_{L\times L} & I_{L\times L}\end{bmatrix}$$

where: $O_{L\times L}$ is *L* – dimensional zero matrix, $I_{L\times L}$ is *L* – dimensional unit matrix

Determinant of Jacobi matrix:

$$|J|=\left|\begin{bmatrix}\frac{\partial h_z^{-1}(y,w)}{\partial w} & \frac{\partial h_z^{-1}(y,w)}{\partial y}\\ O_{L\times L} & I_{L\times L}\end{bmatrix}\right|=\left|\frac{\partial h_z^{-1}(y,w)}{\partial w}\right|=|J_h|$$

Maximum likelihood method

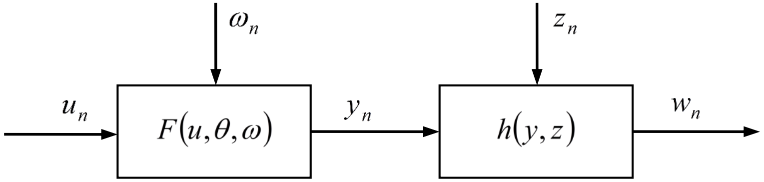
Joint probability density function of random variables (w,y):

$$f_w(w,y,\theta;u)= f_z\left(h_z^{-1}(y,w)\right)f_y(y,\theta;u)\left|J\right| = f_z\left(h_z^{-1}(y,w)\right)f_\omega\left(F_\omega^{-1}(u,\theta,y)\right)\left|J_F\right|\left|J_h\right|$$

Marginal probability density function of random wariable w:

$$f_w(w,\theta;u)=\int_{Y} f_z\left(h_z^{-1}(y,w)\right)f_y(y,\theta;u)\left|J\right|dy = \int_{Y} f_z\left(h_z^{-1}(y,w)\right)f_\omega\left(F_\omega^{-1}(u,\theta,y)\right)\left|J_F\right|\left|J_h\right|dy$$

Likelihood function:



$$L_N(W_N,\theta;U_N)=\prod_{n=1}^{N} f_w(w_n,\theta;u_n)=$$

$$=\prod_{n=1}^{N}\int_{Y} f_z\left(h_z^{-1}(y,w_n)\right)f_\omega\left(F_\omega^{-1}(u_n,\theta,y)\right)\left|J_F\right|\left|J_h\right|dy$$

Estimation algorithm: $\theta_N = \Psi_N(U_N,Y_N) \rightarrow L_N(W_N,\theta_N;U_N)=\max_{\theta\in\Theta} L_N(W_N,\theta;U_N)$

Bayesian method

Assumptions:

Additionally θ is value of random variable $\underline{\theta}$, with probability density function $f_\theta(\theta)$

For a given parametr θ and input u_n the sequence $y_n, n=1, 2, \ldots, N$ are values of random variable $\underline{y}$ under condition $\underline{\theta}=\theta$ and input is equal u .

$w_n, n=1, 2, \ldots, N$ are values of random variable $\underline{w}$ under condition $\underline{\theta}=\theta$

z_n is value of random variable $\underline{z}$, with probability density function $f_z(z)$

Calculations:

$$\underline{w}=h(\underline{y},\underline{z})$$

$$f_w(w|\theta;u)=\int_{Y} f_z\left(h_z^{-1}(y,w)\right)f_y(y|\theta;u)|J|dy=\int_{Y} f_z\left(h_z^{-1}(y,w)\right)f_\omega\left(F_\omega^{-1}(u,\theta,y)\right)|J_F||J_h|dy$$

Bayesian method

A'posteriori probability density function:

$$f'(\theta|W_N;U_N) = \frac{f_\theta(\theta)\prod_{n=1}^{N}\int_{Y} f_z\left(h_z^{-1}(y,w_n)\right) f_\omega\left(F_\omega^{-1}(u_n,\theta,y)\right)|J_F\|J_h|dy}{\int_{\Theta} f_\theta(\theta)\prod_{n=1}^{N}\int_{Y} f_z\left(h_z^{-1}(y,w_n)\right) f_\omega\left(F_\omega^{-1}(u_n,\theta,y)\right)|J_F\|J_h|dy\,d\theta}$$

Conditional risk :

$$r(\overline{\theta},Y_N;U_N) \overset{\text{df}}{=} \underset{\underline{\theta}}{E}\left[L(\underline{\theta},\overline{\theta})\middle|W_N;U_N\right] = \int_{\Theta} L\left(\theta,\overline{\theta}=\overline{\Psi}(U_N,W_N)\right) f'(\theta|W_N;U_N)\,d\theta$$

Estimation algorithm: $\theta_N = \Psi_N(U_N,W_N) \quad \rightarrow \quad r(\theta_N,W_N;U_N) = \min_{\overline{\theta}\in\Theta} r(\overline{\theta},W_N;U_N)$

Thank you for attention