# Computer Science

Jerzy Świątek

# Systems Modelling and Analysis

Choose yourself and new technologies

## L.8. Estimation of plant parameter with noisy measurements

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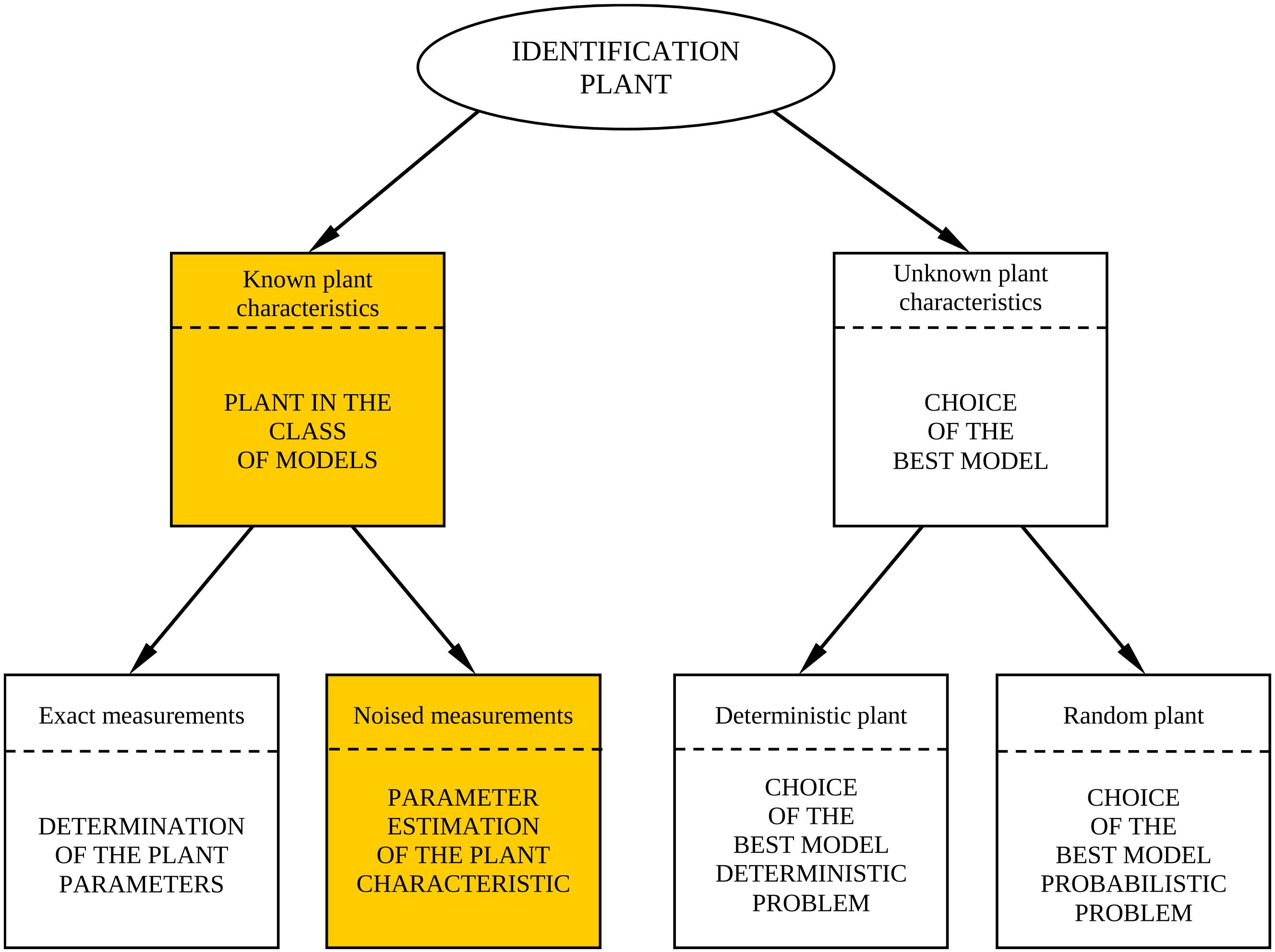



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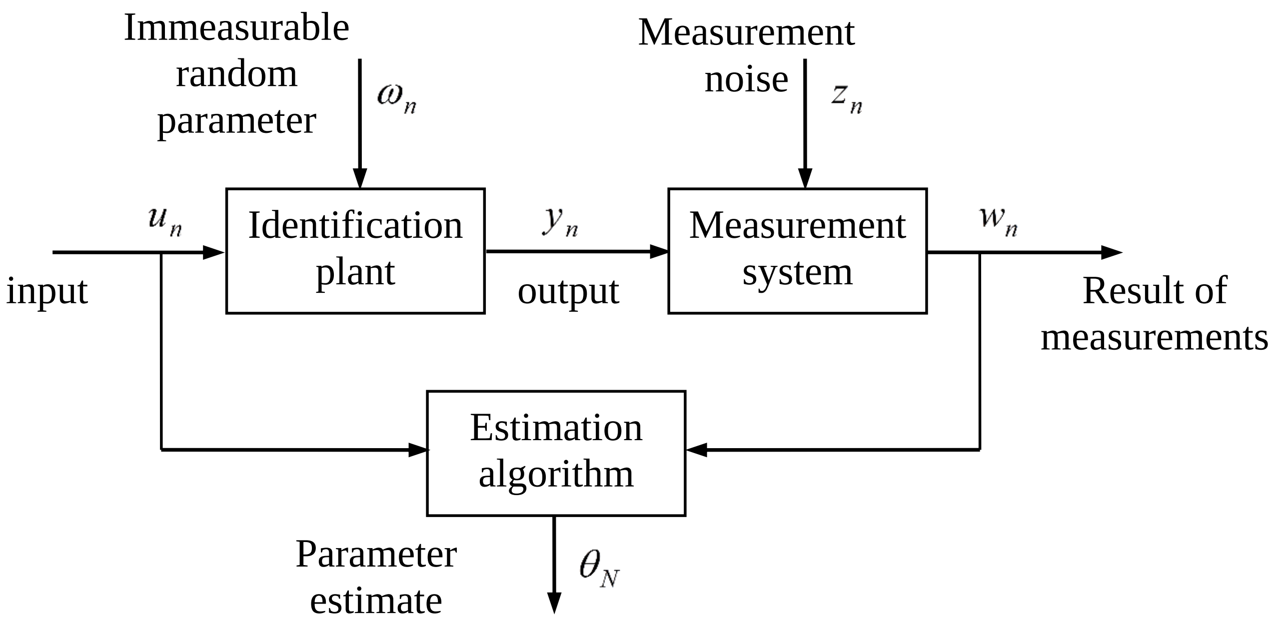
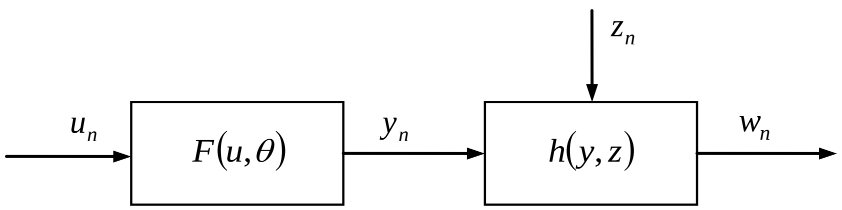
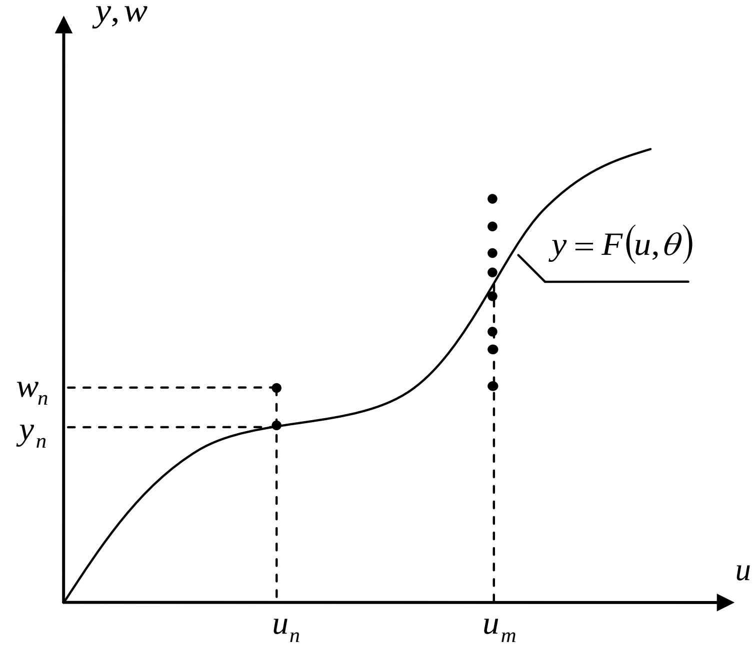
# Plant parameter estimation problem



# Deterministic plant, noised measurements of the plant output

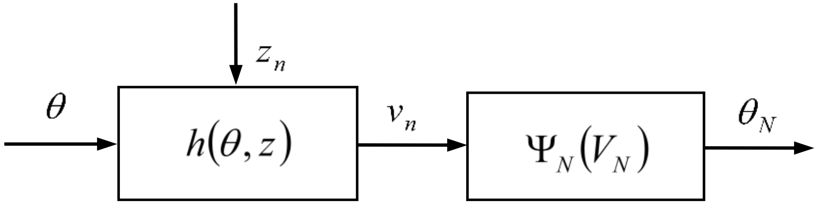
- Noised measurements of the identification plant known static characteristics



# Noised measurements of the physical values

- Problem formulation




<u>Measurement noise:</u>

$z_n$ – value of random variable $\underline{z}$ from the space $\mathsf{Z}$

$f_z(z)$ – probability density function

$\theta$ – observed vector of parameters, value of random variable $\underline{\theta}$, $\theta \in \Theta \subseteq \mathsf{R}^R$

$f_\theta(\theta)$ – probability density function

Measurements: $V_N = [v_1 \quad v_2 \quad \cdots \quad v_N]$

# Noised measurements of the physical values

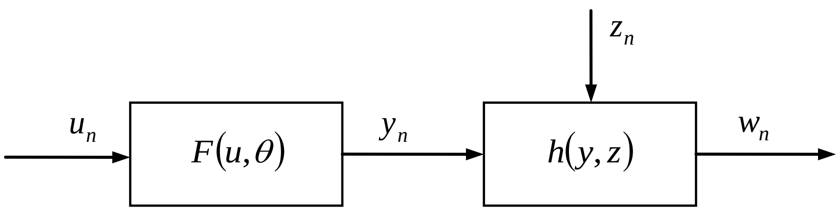
General form of estimation algorithm:

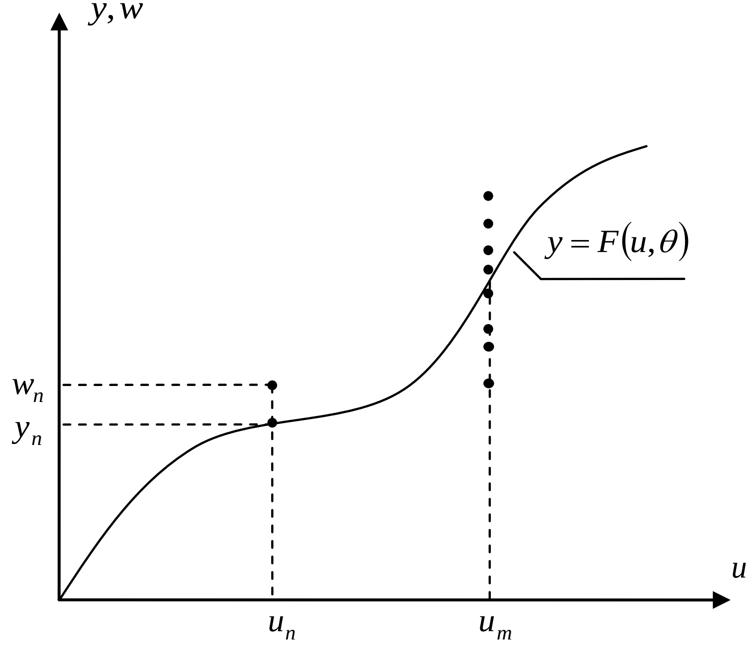
$$\theta_N = \Psi_N(V_N)$$

- Solution:
  - Least square method
  - Maximum likelihood method
  - Bayesian method

# Deterministic plant, noised measurements of the plant output

- Noised measurements of the identification plant known static characteristics

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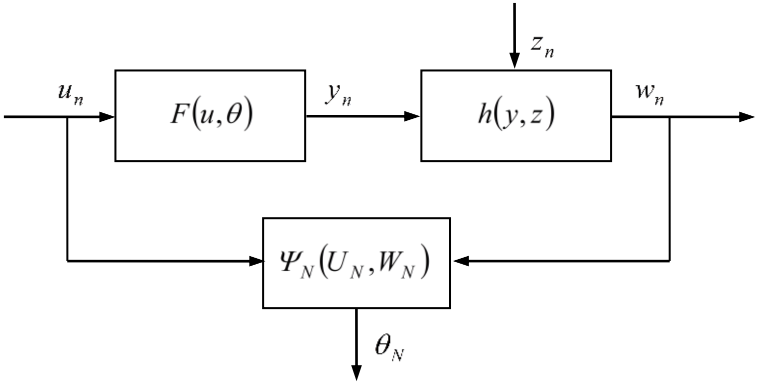
# Deterministic plant, noised measurements of the plant output

- Problem formulation




<u>Measurement system description:</u> $w = h(y,z)$

where: $w \in \mathcal{W}$, $h$ – known one-to-one function

$h : \mathcal{Y} \times \mathcal{Z} \to \mathcal{W}$, $z = h_z^{-1}(y,w)$

$\mathcal{W}$ – measurements domain ( $\dim y = \dim z = L$ )

<u>Measurement noise:</u>

$z_n$ – value of random variable $\underline{z}$ from the space $\mathcal{Z}$   $f_z(z)$ – probability density function

<u>Measurements:</u> $U_N = [u_1 \quad u_2 \quad \cdots \quad u_N]$, $W_N = [w_1 \quad w_2 \quad \cdots \quad w_N]$

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# Deterministic plant, noised measurements of the plant output

General form of estimation algorithm:

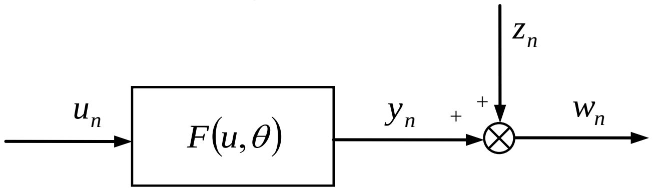
$$\theta_N = \Psi_N\left(U_N, W_N\right)$$

- Solution:
  - Least square method
  - Maximum likelihood method
  - Bayes' method

# Least square method

**(one dimensional case $L=1$)**

Assumptions:




$w = h(y,z) = y + z$ – additive noise

$\underset{\underline{z}}{E}[\underline{z}] = 0$ – expected value of the noise signal is zero

$\underset{\underline{z}}{Var}[\underline{z}] < \infty$ – variance of noise is not infinite

Calculations:

Least square method minimizes empirical variance of noise signal:

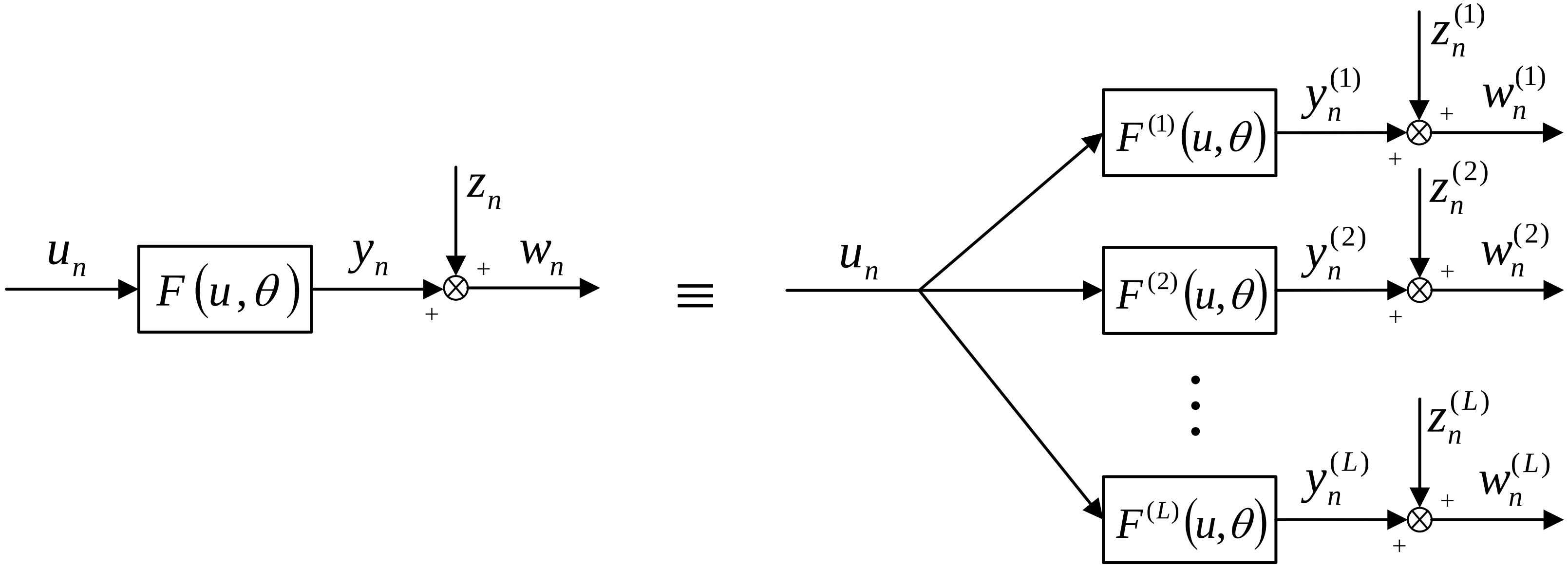
$$Var_{zN}(U_N, W_N, \theta) = \frac{1}{N}\sum_{n=1}^{N}(w_n - y_n)^2 = \frac{1}{N}\sum_{n=1}^{N}(w_n - F(u_n,\theta))^2$$

Estimation algorithm has the form:

$$\theta_N = \Psi_N(U_N, W_N) \quad \rightarrow \quad Var_{zN}(U_N, W_N, \theta_N) = \min_{\theta\in\Theta} Var_{zN}(U_N, W_N, \theta)$$

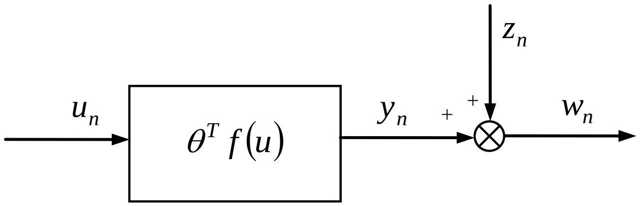
# Least square method

Decomposition of $L$ – dimensional identification plant into $L$ single output plants:



# Least square method

- Example



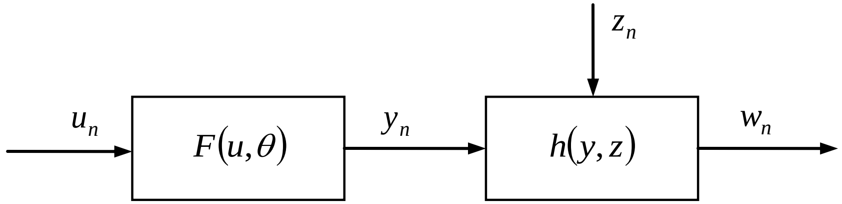

Empirical variance of noise signal:

$$Var_{zN}\left(U_N, W_N, \theta\right) = \frac{1}{N}\sum_{n=1}^{N}\left(w_n - y_n\right)^2 = \frac{1}{N}\sum_{n=1}^{N}\left(w_n - \theta^T f\left(u_n\right)\right)^2$$

Estimation algorithm:

$$\theta_N = \Psi_N\left(U_N, W_N\right) = \left[\sum_{n=1}^{N} f\left(u_n\right) f^T\left(u_n\right)\right]^{-1} \sum_{n=1}^{N} w_n f\left(u_n\right)$$

# Maximum likelihood method




Assumptions:

$\underline{w} = h(y, \underline{z})$ – measurement system is described by any one-to-one invertible function

$z_n$ – independent value of random variable $\underline{z}$ with known probability density function $f_z(z)$

For a given $u_n, n = 1, 2, \ldots, N$ noised output is measured $w_n, n = 1, 2, \ldots, N$

Calculations:

Taking into account plant description $\underline{w} = h(F(u,\theta), \underline{z})$, probability density function of random variable $\underline{w} - f_w(w,\theta;u)$ with parameters $\theta$ has the form: $f_w(w,\theta;u) = f_z\left(h_z^{-1}(F(u,\theta), w)\right) \cdot |J_h|$

where $J_h$ is Jacobi matrix: $J_h = \dfrac{\partial h_z^{-1}(y,w)}{\partial w}$

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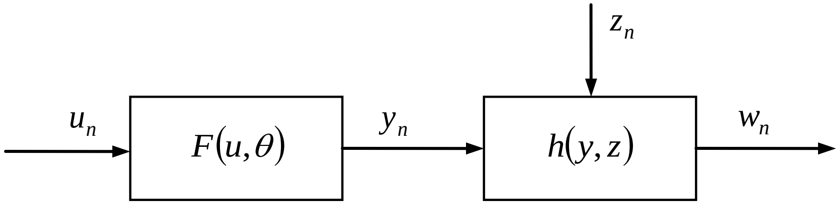
# Maximum likelihood method




Likelihood function:

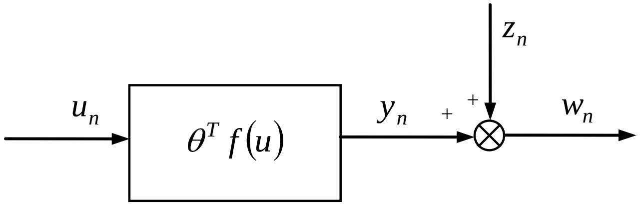
$$L_N(W_N,\theta;U_N)=\prod_{n=1}^{N} f_w(w_n,\theta;u_n)=\prod_{n=1}^{N} f_z\left(h_z^{-1}\left(F(u_n,\theta),w_n\right)\right)\left|J_h\right|$$

Estimation algorithm:

$$\theta_N=\Psi_N(U_N,W_N) \quad \rightarrow \quad L_N(W_N,\theta_N;U_N)=\max_{\theta\in\Theta} L_N(W_N,\theta;U_N)$$

# Maximum likelihood method

- Example 1




Noise description: $$f_z(z) = \frac{1}{\sigma_z \sqrt{2\pi}} \exp\left[ -\frac{(z - m_z)^2}{2\sigma_z^2} \right]$$

Measurement system description: $$w = h(y, z) = y + z$$

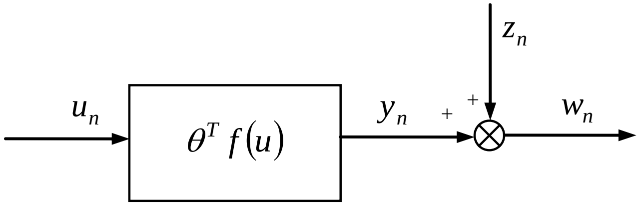
$$z = h_z^{-1}(F(u, \theta), w) = w - \theta^T f(u)$$

Jacobi matrix of the inverse transform :

$$J_h = \frac{\partial h_z^{-1}(F(u, \theta), w)}{\partial w} = \frac{d}{dw}\left(w - \theta^T f(u)\right) = 1$$

# Maximum likelihood method

- Example 1




Probability density function : $f_w(w,\theta;u)=\frac{1}{\sigma_z\sqrt{2\pi}}\exp\left[-\frac{\left(w-\theta^T f(u)-m_z\right)^2}{2\sigma_z^2}\right]\cdot|1|$

Likelihood function : $L_N(W_N,\theta;U_N)=\prod_{n=1}^{N}\frac{1}{\sigma_z\sqrt{2\pi}}\exp\left[-\frac{\left(w_n-\theta^T f(u_n)-m_z\right)^2}{2\sigma_z^2}\right]$

$$L_N(W_N,\theta;U_N)=\left(\frac{1}{\sigma_z\sqrt{2\pi}}\right)^N\exp\left[\sum_{n=1}^{N}-\frac{\left(w_n-\theta^T f(u_n)-m_z\right)^2}{2\sigma_z^2}\right]$$

Estimation algorithm:

$$\theta_N=\Psi_N(U_N,W_N)=\left[\sum_{n=1}^{N}f(u_n)\,f^T(u_n)\right]^{-1}\sum_{n=1}^{N}(w_n-m_z)f(u_n)$$

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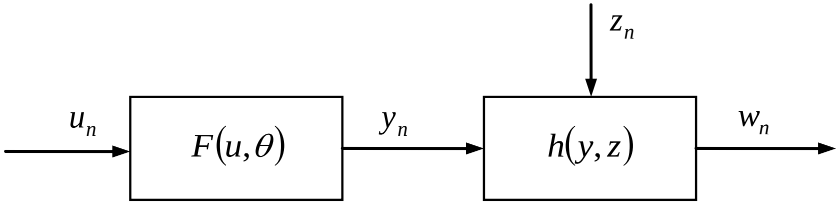
# Maximum likelihood method

- Example 2




Plant characteristic: $y = F(u,\theta) = \theta\, u \qquad (\theta > 0)$

Noise description: $f_z(z) = \begin{cases} 1 & \text{for} \quad z \in [0,1] \\ 0 & \text{for} \quad z \notin [0,1] \end{cases}$

Additional assumption: $\forall\, n = 1, 2, \ldots, N \quad u_n > 0$

Measurement system description: $w = h(y,z) = h(F(u,\theta), z) = \theta\; u\; z$

$$z = h_z^{-1}(F(u,\theta), w) = \frac{w}{\theta\, u}$$

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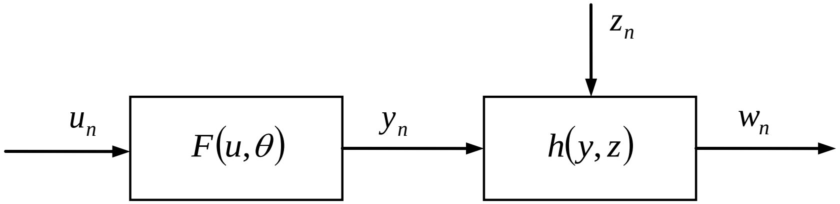
# Maximum likelihood method

- Example 2



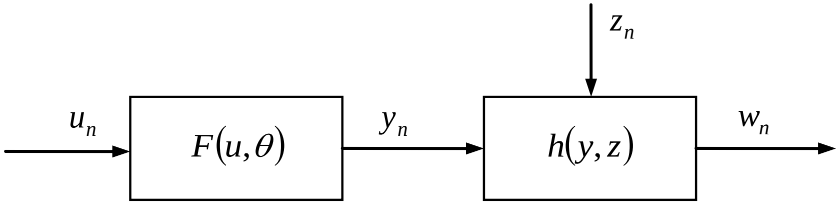

Jacobi matrix: $$J_h = \frac{\partial h_z^{-1}(F(u,\theta),w)}{\partial w} = \frac{d}{dw}\left(\frac{w}{\theta u}\right) = \frac{1}{\theta u}$$

Probability density function of the observed value:

$$f_w(w,\theta;u) = \begin{cases} \dfrac{1}{\theta u} & \text{for} \quad \dfrac{w}{\theta u} \in [0,1] \\ 0 & \text{for} \quad \dfrac{w}{\theta u} \notin [0,1] \end{cases}$$

# Maximum likelihood method

- Example 2



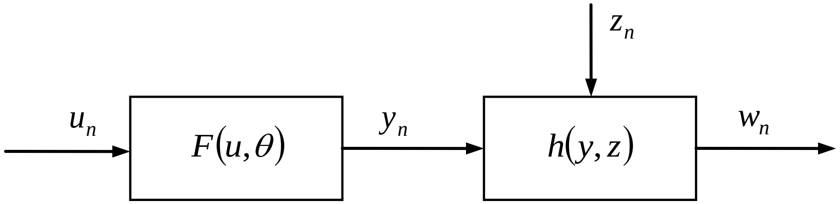

Likelihood function :

$$L_N(W_N,\theta;U_N)=\begin{cases}\dfrac{1}{\theta\prod\limits_{n=1}^{N}u_n} & \text{for} \quad \forall n=1,2,\ldots,N \quad \dfrac{w_n}{u_n}\in[0,\theta] \\ 0 & \text{for} \quad \exists n=1,2,\ldots,N \quad \dfrac{w_n}{u_n}\notin[0,\theta]\end{cases}$$

$$L_N(W_N,\theta;U_N)=\begin{cases}\dfrac{1}{\theta\prod\limits_{n=1}^{N}u_n} & \text{for} \quad \theta\geq\max\limits_{1\leq n\leq N}\left\{\dfrac{w_n}{u_n}\right\} \\ 0 & \text{for} \quad \theta<\max\limits_{1\leq n\leq N}\left\{\dfrac{w_n}{u_n}\right\}\end{cases}$$

# Maximum likelihood method

- Example 2



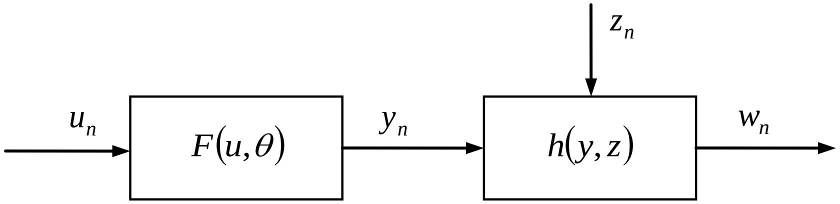

Estimation algorithm:

$$\theta_N = \Psi_N(U_N, W_N) = \max_{1\le n\le N}\left\{\frac{w_n}{u_n}\right\}$$

Interpretation:

$$\theta_N = \max_{1\le n\le N}\left\{\frac{w_n}{u_n}\right\} = \max_{1\le n\le N}\left\{\frac{\theta\, u_n\, z_n}{u_n}\right\} = \theta \max_{1\le n\le N}\{z_n\}$$

# Bayesian method




Assumptions:

$\underline{w} = h(y, \underline{z})$ – measurement system is described by any one-to-one invertible function

$z_n$ – independent value of random variable $\underline{z}$ with known probability density function $f_z(z)$

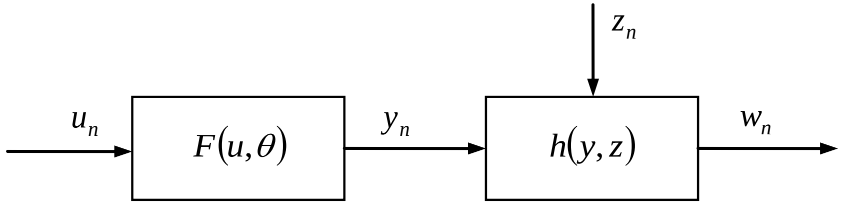
Mathematical formulas describing probability density functions $f_z(z)$ and $f_\theta(\theta)$ are given.

The loss function $L(\theta, \bar{\theta})$ is defined.

For a given $u_n, n = 1, 2, \ldots, N$ noised output is measured $w_n, n = 1, 2, \ldots, N$

$\theta$ – plant parameters vector, value of random variable $\underline{\theta}$, $\theta \in \Theta \subseteq \mathrm{R}^R$

# Bayesian method




<u>Calculations:</u>

Taking into account plant description: $\underline{w} = h(F(u,\theta), \underline{z})$

Risk: $R(\overline{\Psi}) \overset{\mathrm{df}}{=} \underset{\underline{\theta},\underline{W_N}}{E}\left[L\left(\underline{\theta}, \overline{\underline{\theta}} = \overline{\Psi}(U_N, \underline{W_N})\right)\right] = \int\limits_{W_N}\int\limits_{\Theta} L\left(\theta, \overline{\Psi}(U_N, W_N)\right) f(\theta, W_N; U_N)\, d\theta\, dW_N$

where $f(\theta, W_N; U_N)$ is joint probability density function:

$$f(\theta, W_N; U_N) = f'(\theta | W_N; U_N) f''(W_N; U_N)$$

The problem: $\Psi_N \rightarrow R(\Psi_N) = \min_{\overline{\Psi}} R(\overline{\Psi})$

# Bayesian method

Joint probability density function: $f\left(\theta, W_N; U_N\right) = f'\left(\theta \middle| W_N; U_N\right) f''\left(W_N; U_N\right)$

where: $$f'\left(\theta \middle| W_N; U_N\right) = \frac{f_\theta(\theta)\, f_{WN}\left(W_N \middle| \theta; U_N\right)}{f''\left(W_N; U_N\right)} = \frac{f_\theta(\theta)\, f_{WN}\left(W_N \middle| \theta; U_N\right)}{\int\limits_\Theta f_\theta(\theta)\, f_{WN}\left(W_N \middle| \theta; U_N\right) d\theta}$$

$$f_{WN}\left(W_N \middle| \theta; U_N\right) = \prod_{n=1}^{N} f_w\left(w_n \middle| \theta; u_n\right)$$

Conditional probability density function of observed value: $f_w\left(w \middle| \theta; u\right) = f_z\left(h_z^{-1}\left(F(u,\theta), w\right)\right)\left|J_h\right|$

$$f'\left(\theta \middle| W_N; U_N\right) = \frac{f_\theta(\theta) \prod_{n=1}^{N} f_z\left(h_z^{-1}\left(F(u_n,\theta), w_n\right)\right)\left|J_h\right|}{\int\limits_\Theta f_\theta(\theta) \prod_{n=1}^{N} f_z\left(h_z^{-1}\left(F(u_n,\theta), w_n\right)\right)\left|J_h\right| d\theta} \quad \alpha \quad f_\theta(\theta) \prod_{n=1}^{N} f_z\left(h_z^{-1}\left(F(u_n,\theta), w_n\right)\right)\left|J_h\right|$$

# Bayesian method

Risk:

$$R(\overline{\Psi}) = \int_{W_N}\int_{\Theta} L\big(\theta,\overline{\Psi}(U_N,W_N)\big)\, f'(\theta|W_N;U_N)\,d\theta\; f''(W_N;U_N)\,dW_N$$

Conditional risk:

$$r(\overline{\theta},W_N;U_N) \stackrel{\mathrm{df}}{=} \underset{\underline{\theta}}{E}\big[L(\underline{\theta},\overline{\theta})\big|W_N;U_N\big] = \int_{\Theta} L\big(\theta,\overline{\theta}=\overline{\Psi}(U_N,W_N)\big)\, f'(\theta|W_N;U_N)\,d\theta$$

Estimation algorithm:

$$\theta_N = \Psi_N(U_N,W_N) \quad \rightarrow \quad r(\theta_N,W_N;U_N) = \min_{\overline{\theta}\in\Theta} r(\overline{\theta},W_N;U_N)$$

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# Bayesian method

- Mean a posteriori value method

Special case of the loss function: $L(\theta,\overline{\theta})=\left[\theta-\overline{\theta}\right]^T\left[\theta-\overline{\theta}\right]$

Conditional risk: $r\left(\overline{\theta},W_N;U_N\right)\overset{df}{=}\underset{\underline{\theta}}{E}\left[\left[\underline{\theta}-\overline{\theta}\right]^T\left[\underline{\theta}-\overline{\theta}\right]\middle|W_N;U_N\right]=$

$$=\underset{\underline{\theta}}{E}\left[\underline{\theta}^T\,\underline{\theta}\middle|W_N;U_N\right]-2\overline{\theta}\,\underset{\underline{\theta}}{E}\left[\underline{\theta}^T\middle|W_N;U_N\right]+\overline{\theta}^T\overline{\theta}$$

$$\underset{\overline{\theta}}{\operatorname{grad}}\,r\left(\overline{\theta},W_N;U_N\right)\Big|_{\overline{\theta}=\theta_N}=-2\,\underset{\underline{\theta}}{E}\left[\underline{\theta}\middle|W_N;U_N\right]+2\theta_N=0_N$$

Estimation algorithm:

$$\theta_N=\Psi_N\left(U_N,W_N\right)=\underset{\underline{\theta}}{E}\left[\underline{\theta}\middle|W_N;U_n\right]=\int_{\Theta}\theta f'\left(\theta\middle|W_N;U_N\right)d\theta$$

# Bayesian method

- Maximum a posteriori probability

Special case of the loss function: $L(\theta,\bar{\theta}) = -\delta(\theta-\bar{\theta})$

Conditional risk: $r(\bar{\theta},W_N;U_N) = -\int\limits_{\Theta}\delta(\theta-\bar{\theta})f'(\theta|W_N;U_N)\,d\theta = -f'(\bar{\theta}|W_N;U_N)$

$$\theta_N = \Psi_N(U_N,W_N) \quad \rightarrow \quad f'(\theta_N|W_N;U_N) = \max_{\bar{\theta}\in\Theta} f'(\bar{\theta}|W_N;U_N)$$

$$f'(\theta_N|W_N;U_N) \quad \alpha \quad f_\theta(\theta_N)\prod_{n=1}^{N} f_z\left(h_z^{-1}(F(u_n,\theta_N),w_n)\right)|J_h|$$

$\theta_N = \Psi_N(U_N,W_N)$ such that:

$$f_\theta(\theta_N)\prod_{n=1}^{N} f_z\left(h_z^{-1}(F(u_n,\theta_N),w_n)\right)|J_h| = \max_{\bar{\theta}\in\Theta} f_\theta(\bar{\theta})\prod_{n=1}^{N} f_z\left(h_z^{-1}(F(u_n,\bar{\theta}),w_n)\right)|J_h|$$

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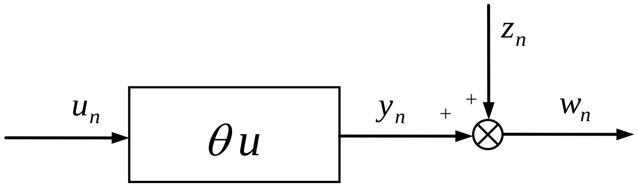
# Bayesian method

- Example




Plant characteristic: $y = F(u, \theta) = \theta\, u$

Noise description: $f_z(z) = \frac{1}{\sigma_z \sqrt{2\pi}} \exp\left[ -\frac{z^2}{2\sigma_z^2} \right]$

A priori distribution: $f_\theta(\theta) = \frac{1}{\sigma_\theta \sqrt{2\pi}} \exp\left[ -\frac{(\theta - m_\theta)^2}{2\sigma_\theta^2} \right]$

Measurement system description: $w = h(y, z) = y + z, \quad z = h_z^{-1}(F(u,\theta), w) = w - \theta\, u$

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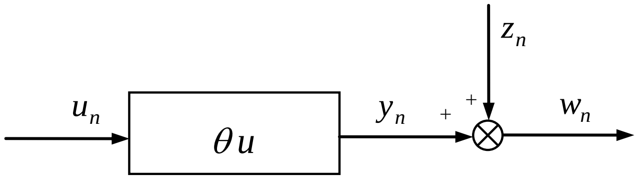
# Bayesian method

- Example




Jacobi matrix: $$J_h = \frac{\partial h_z^{-1}(F(u,\theta),w)}{\partial w} = \frac{d}{dw}(w-\theta u)=1$$

Probability density function of the observed value: $$f_w(w|\theta;u) = \frac{1}{\sigma_z\sqrt{2\pi}}\exp\left[-\frac{(w-\theta u)^2}{2\sigma_z^2}\right]\cdot|1|$$

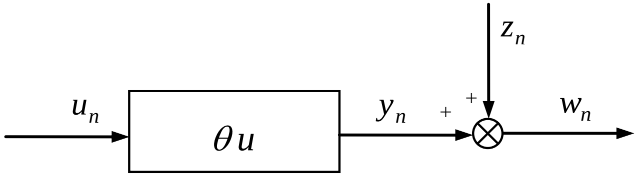
A posteriori probability density function:

$$f'(\theta|W_N;U_N) \quad \alpha \quad f_\theta(\bar{\theta})\prod_{n=1}^{N} f_z\left(h_z^{-1}(F(u_n,\theta),w_n)\right)|J_h| =$$

$$= \frac{1}{\sigma_\theta\sqrt{2\pi}}\exp\left[-\frac{(\bar{\theta}-m_\theta)^2}{2\sigma_\theta^2}\right]\prod_{n=1}^{N}\frac{1}{\sigma_z\sqrt{2\pi}}\exp\left[-\frac{(w_n-\bar{\theta}\,u_n)^2}{2\sigma_z^2}\right]$$

# Bayesian method

- Example




A posteriori probability density function after transformation:

$$f'(\theta|V_N) \quad \alpha \quad f_\theta(\overline{\theta}) \prod_{n=1}^{N} f_z\left(h_z^{-1}\left(F\left(u_n,\overline{\theta}\right), w_n\right)\right)|J_h| =$$

$$= \frac{1}{\sigma_\theta\sqrt{2\pi}}\left(\frac{1}{\sigma_z\sqrt{2\pi}}\right)^N \exp\left[-\frac{\left(\overline{\theta}-m_\theta\right)^2}{2\sigma_\theta^2} - \sum_{n=1}^{N}\frac{\left(w_n-\overline{\theta}\,u_n\right)^2}{2\sigma_z^2}\right]$$

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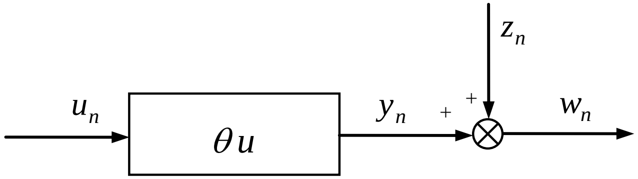


Estimation algorithm:

$$\theta_N = \Psi_N(U_N, W_N) = \frac{m_\theta + \left(\frac{\sigma_\theta}{\sigma_z}\right)^2 \sum_{n=1}^{N} w_n u_n}{1 + \left(\frac{\sigma_\theta}{\sigma_z}\right)^2 \sum_{n=1}^{N} u_n^2}$$

Discussion:

1° $N$ – small number

$(\sigma_z >> \sigma_\theta)$ – poor measurements

$$\theta_N \approx m_\theta$$

2° $N \to \infty$

$(\sigma_z << \sigma_\theta)$ – good measurements

$$\theta_N \approx \frac{\sum_{n=1}^{N} w_n u_n}{\sum_{n=1}^{N} u_n^2}$$


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# Thank you for attention