

Computer Science

Jerzy Świątek

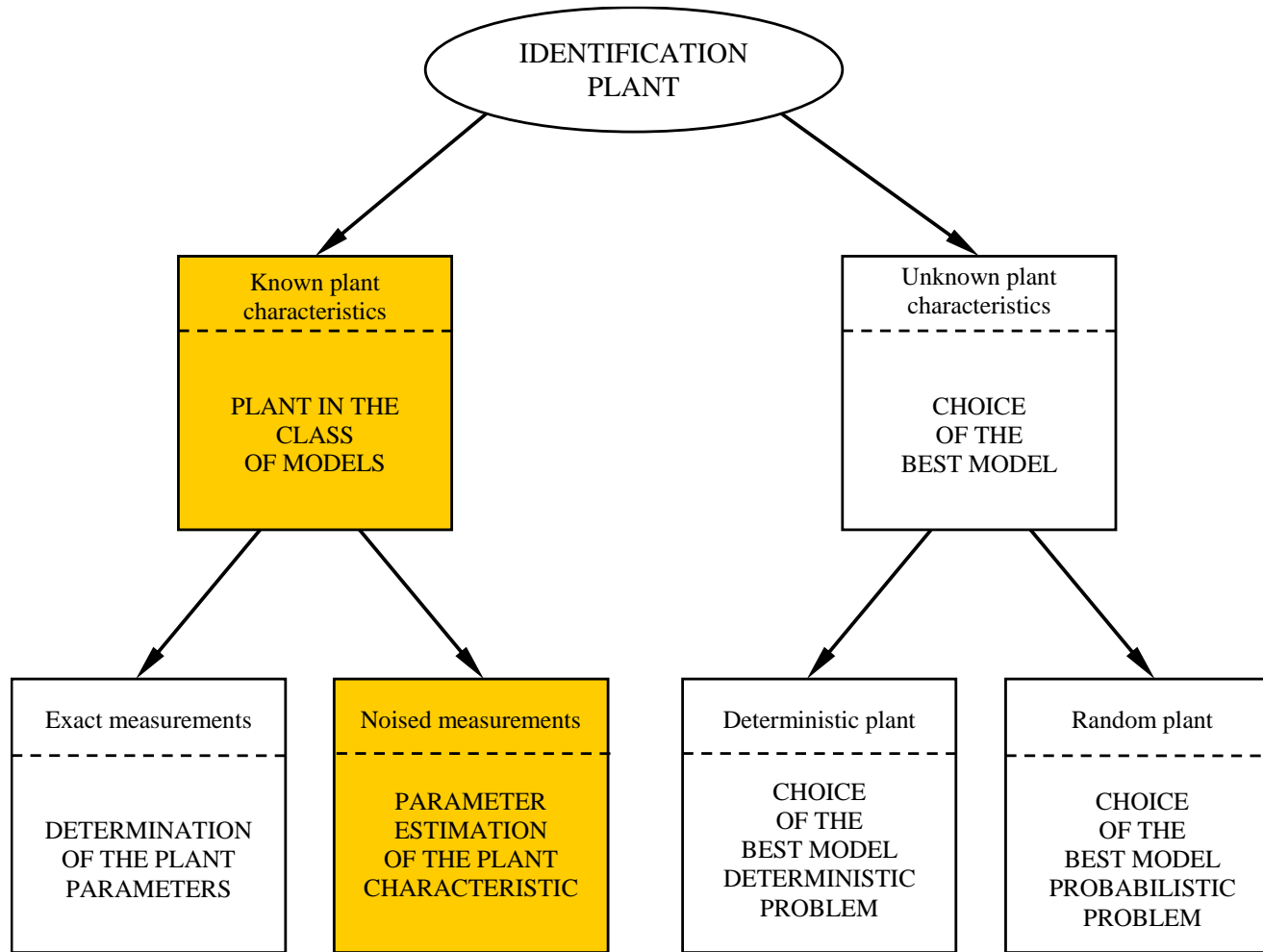
Systems Modelling and Analysis

Choose yourself and new technologies

L.7. Noised measurements of the physical values



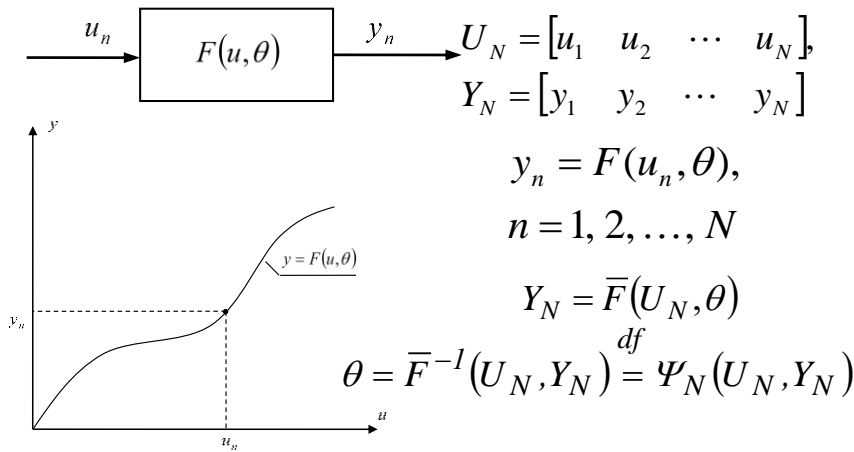
Project co-financed from the EU European Social Fund



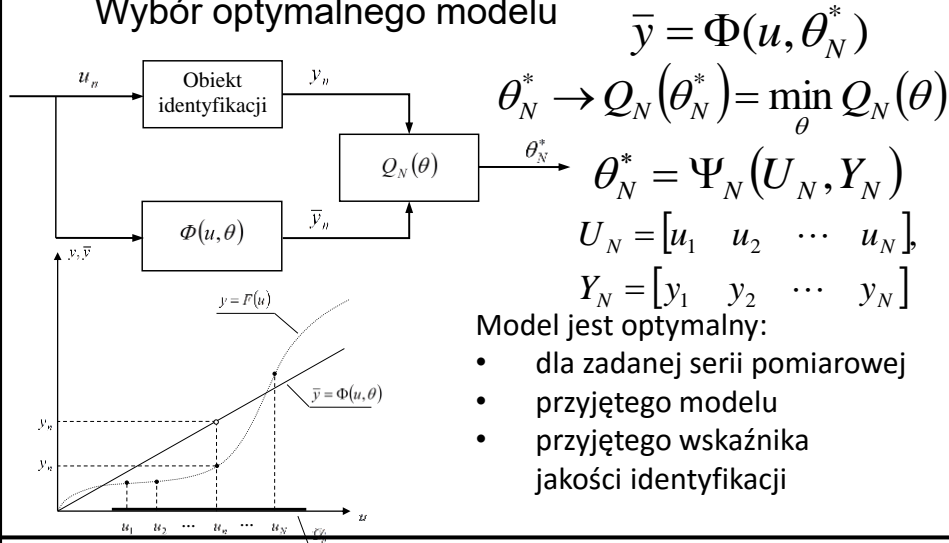


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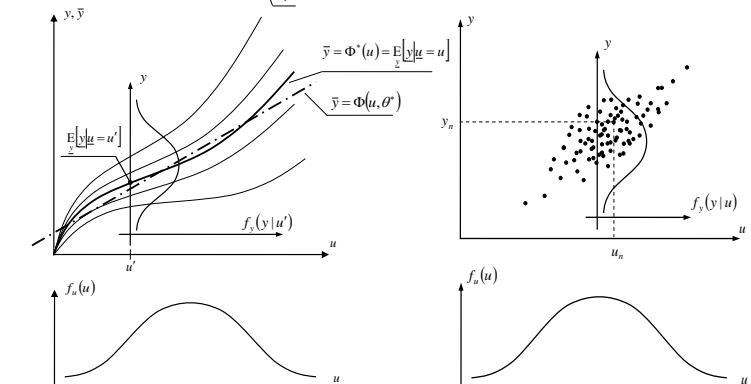
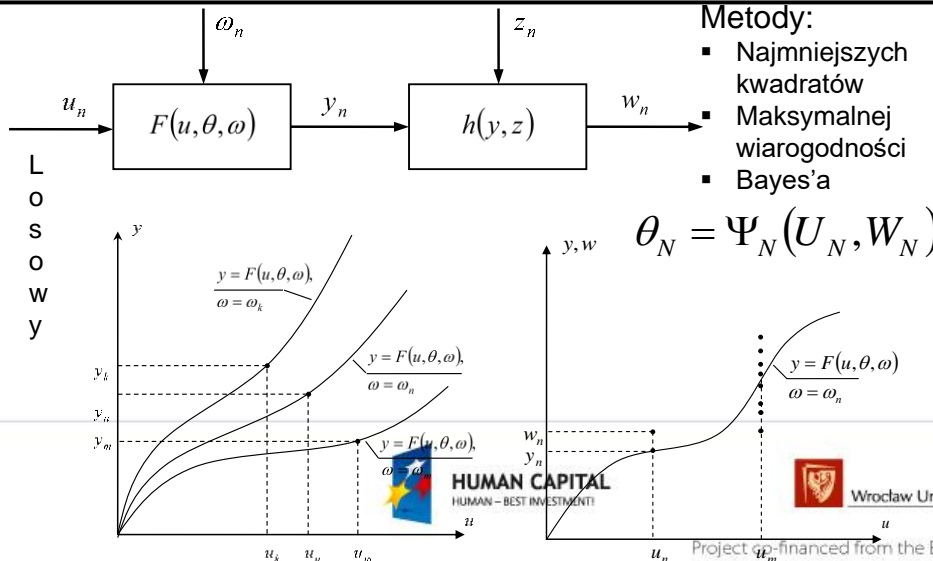
Obiekt w klasie modeli



Wybór optymalnego modelu



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Pełna informacja

- Regresja I rodzaju
- Regresja II rodzaju

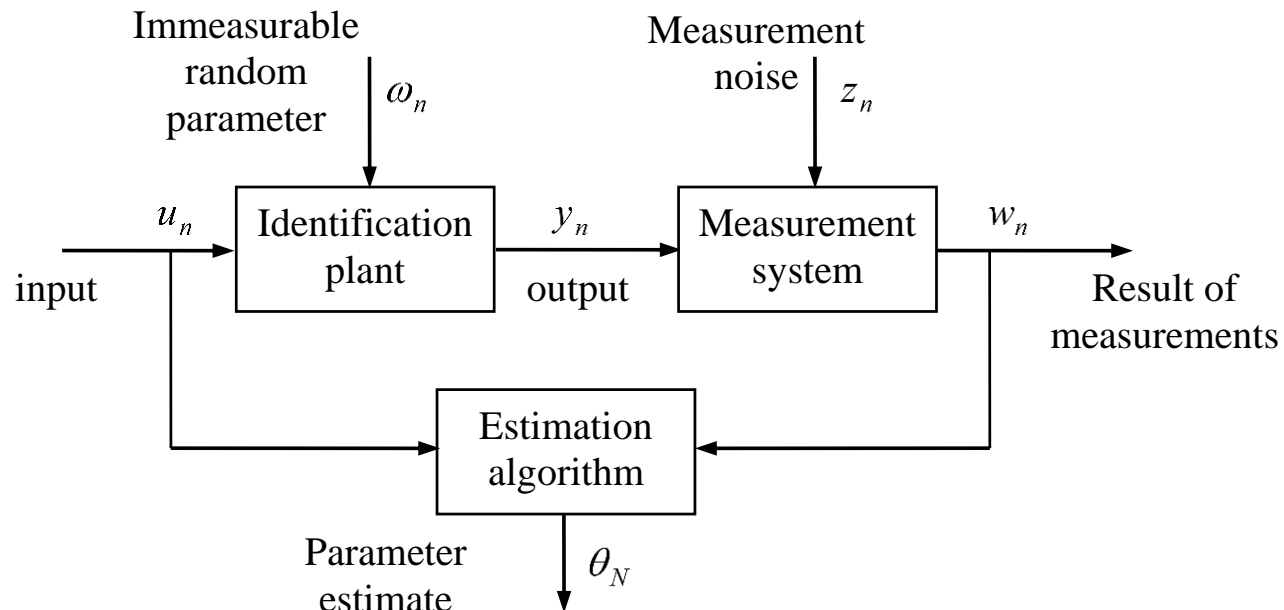
Niepełna informacja

- Estymacja wskaźnika jakości
- Estymacja parametrów rozkładu
- Estymacja rozkładu





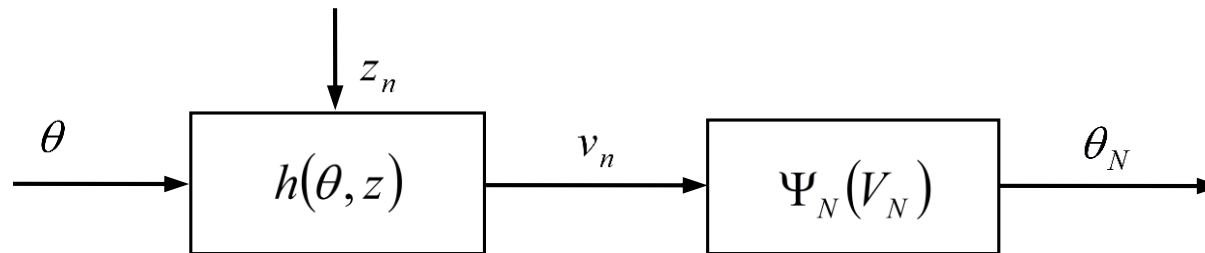
Plant parameter estimation problem





Plant parameter estimation problem

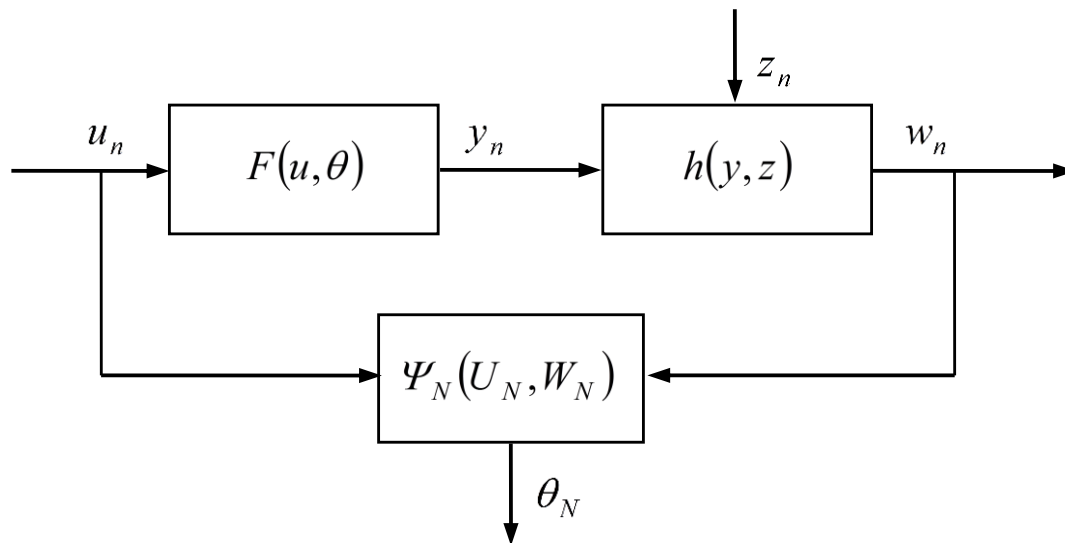
- Noised measurements of the physical values





Plant parameter estimation problem

- Deterministic plant, noised measurements of the plant output



where:

$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N]$$

$$W_N = [w_1 \quad w_2 \quad \cdots \quad w_N]$$

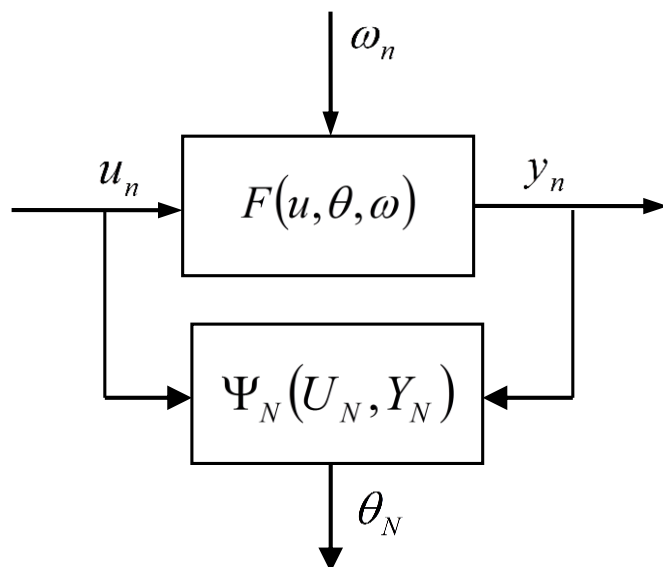
Ψ_N – estimation algorithm

θ_N – estimate of θ



Plant parameter estimation problem

- Immeasurable random plant parameter



where:

$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N]$$

$$Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$$

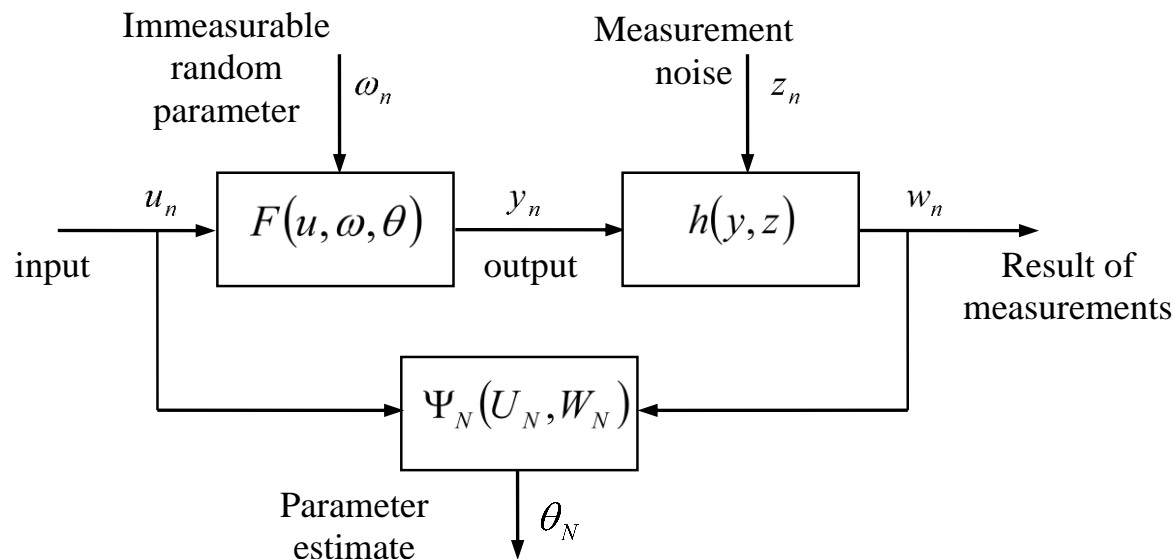
Ψ_N – estimation algorithm

θ_N – estimate of θ



Plant parameter estimation problem

- Immeasurable random plant parameter and noised measurements of the plant output

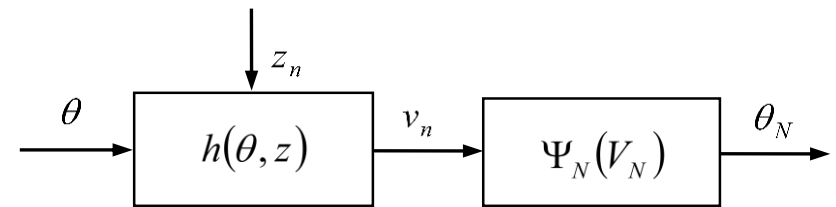




Noised measurements of the physical values

- Problem formulation

Measurement system description: $v = h(\theta, z)$



where: $v \in V$, h – known one-to-one function

$$h: \Theta \times Z \rightarrow V, \quad z = h_z^{-1}(\theta, v)$$

examples of h : $v = h(\theta, z) = \theta + z$

$$v = h(\theta, z) = \theta \cdot z$$

W – measurements domain ($\dim \theta = \dim z = L$)



Noised measurements of the physical values

- Problem formulation

Measurement noise:

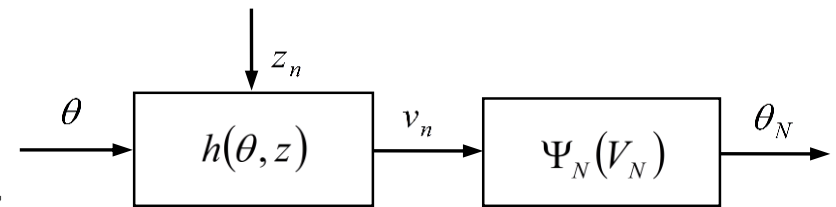
z_n – value of random variable \underline{z} from the space Z

$f_z(z)$ – probability density function

θ – observed vector of parameters, value of random variable $\underline{\theta}$, $\theta \in \Theta \subseteq \mathbb{R}^R$

$f_\theta(\theta)$ – probability density function

Measurements: $V_N = [v_1 \quad v_2 \quad \dots \quad v_N]$





Noised measurements of the physical values

General form of estimation algorithm:

$$\theta_N = \Psi_N(V_N)$$

- Solution:
 - Least square method
 - Maximum likelihood method
 - Bayesian method



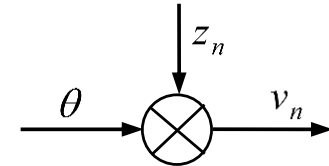
Least square method

Assumptions:

$$v = h(\theta, z) = \theta + z \quad - \text{additive noise}$$

$$E[\underline{z}] = 0 \quad - \text{expected value of the noise signal is zero}$$

$$\text{Var}[\underline{z}] < \infty \quad - \text{variance of the noise is not infinite}$$



Calculations:

Least square method minimizes variance of noise signal:

$$\text{Var}_{zN}(V_N, \theta) = \frac{1}{N} \sum_{n=1}^N (v_n - \theta)^2$$

Estimation algorithm has the form:

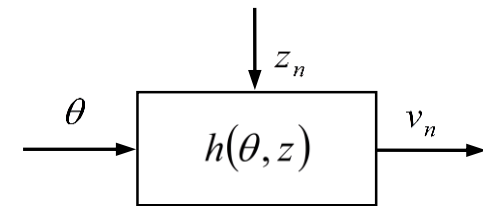
$$\theta_N = \Psi_N(V_N) \rightarrow \text{Var}_{zN}(V_N, \theta_N) = \min_{\theta \in \Theta} \text{Var}_{zN}(V_N, \theta)$$

Estimation algorithm:

$$\theta_N = \frac{1}{N} \sum_{n=1}^N v_n$$



Maximum likelihood method



Assumptions:

$\underline{v} = h(\underline{\theta}, \underline{z})$ – measurement system is described by any one-to-one invertible function

Mathematical formula describing probability density function $f_z(z)$ is given.

Calculations:

Probability density function of observed value \underline{v} with unknown parameter:

$f_v(v, \theta) = f_z(h^{-1}(\theta, v)) \cdot |J_h|$, where J_h is Jacobi matrix of the inverse transformation.

Likelihood function has the form:

$$L_N(\underline{V}_N, \theta) = \prod_{n=1}^N f_v(v_n, \theta) = \prod_{n=1}^N f_z(h_z^{-1}(\theta, v_n)) |J_h|, \quad \text{where: } J_h = \frac{\partial h_z^{-1}(\theta, v)}{\partial v}$$



Maximum likelihood method

Estimation algorithm has the form:

$$\theta_N = \Psi_N(V_N) \rightarrow L_N(V_N, \theta_N) = \max_{\theta \in \Theta} L_N(V_N, \theta)$$



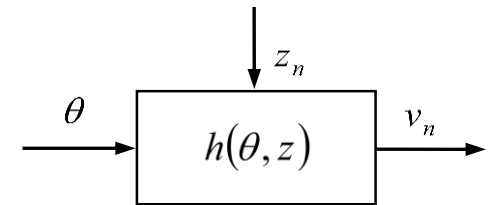
Maximum likelihood method

- Example

Noise description:
$$f_z(z) = \begin{cases} 1 & \text{for } z \in [0, 1] \\ 0 & \text{for } z \notin [0, 1] \end{cases}$$

Measurement system description:
$$v = h(\theta, z) = \theta z \quad (\theta > 0)$$

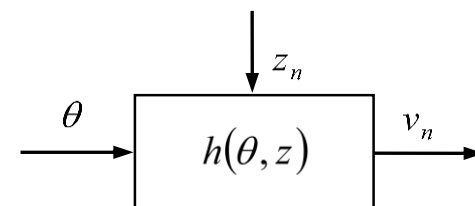
$$z = h_z^{-1}(\theta, v) = \frac{v}{\theta}$$





Maximum likelihood method

- Example



Jacobi matrix:

$$J_h = \frac{\partial h_z^{-1}(\theta, v)}{\partial v} = \frac{d}{dv} \left(\frac{v}{\theta} \right) = \frac{1}{\theta}$$

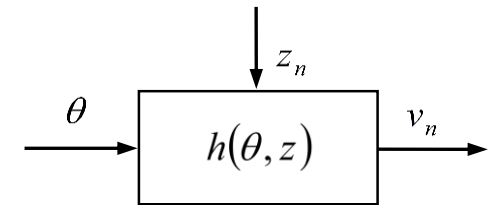
Probability density function of the observed value:

$$f_v(v, \theta) = \begin{cases} \frac{1}{\theta} & \text{for } \frac{v}{\theta} \in [0, 1] \\ 0 & \text{for } \frac{v}{\theta} \notin [0, 1] \end{cases}$$



Maximum likelihood method

- Example



Likelihood function :

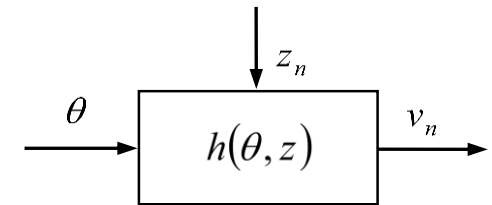
$$L_N(v_N, \theta) = \begin{cases} \frac{1}{\theta^N} & \text{for } \forall n = 1, 2, \dots, N \quad v_n \in [0, \theta] \\ 0 & \text{for } \exists n = 1, 2, \dots, N \quad v_n \notin [0, \theta] \end{cases}$$

$$L_N(W_N, \theta; U_N) = \begin{cases} \frac{1}{\theta^N} & \text{for } \theta \geq \max_{1 \leq n \leq N} \{v_n\} \\ 0 & \text{for } \theta < \max_{1 \leq n \leq N} \{v_n\} \end{cases}$$



Maximum likelihood method

- Example



Estimation algorithm:

$$\theta_N = \Psi_N(V_N) = \max_{1 \leq n \leq N} \{v_n\}$$

Interpretation:

$$\theta_N = \max_{1 \leq n \leq N} \{v_n\} = \max_{1 \leq n \leq N} \{\theta z_n\} = \theta \max_{1 \leq n \leq N} \{z_n\}$$



Bayesian method

Assumptions:

$\underline{v} = h(\theta, \underline{z})$ – measurement system is described by any one-to-one invertible function
 Mathematical formulas describing probability density functions $f_z(\underline{z})$ and $f_\theta(\theta)$ are given.
 The loss function $L(\theta, \bar{\theta})$ is defined, where $\bar{\theta}$ is estimated value of unknown parameter.

Calculations:

$$\text{Risk: } R(\bar{\Psi}) \stackrel{df}{=} E_{\theta, \underline{V}_N} [L(\theta, \bar{\theta} = \bar{\Psi}(\underline{V}_N))] = \int \int_{\underline{V}_N \Theta} L(\theta, \bar{\Psi}(\underline{V}_N)) f(\theta, \underline{V}_N) d\theta d\underline{V}_N$$

where $f(\theta, \underline{V}_N)$ is joint probability density function:

$$f(\theta, \underline{V}_N) = f'(\theta | \underline{V}_N) f''(\underline{V}_N)$$

where f' is conditional probability density function and f'' is marginal probability density function



Bayesian method

The problem: $\Psi_N \rightarrow R(\Psi_N) = \min_{\bar{\Psi}} R(\bar{\Psi})$

$$R(\bar{\Psi}) = \int \int_{V_N \ominus} L(\theta, \bar{\Psi}(V_N)) f'(\theta|V_N) d\theta f''(V_N) dV_N$$

$$r(\bar{\theta}, V_N) \stackrel{\text{df}}{=} E_{\underline{\theta}}[L(\underline{\theta}, \bar{\theta})|V_N] = \int_{\ominus} L(\theta, \bar{\theta} = \bar{\Psi}(V_N)) f'(\theta|V_N) d\theta$$

r – conditional risk



where:





Bayesian method

The problem is reduced to the equivalent one:

$$\theta_N = \Psi_N(V_N) \rightarrow r(\theta_N, V_N) = \min_{\theta \in \Theta} r(\bar{\theta}, V_N)$$

$$f'(\theta|V_N) = \frac{f_\theta(\theta) \prod_{n=1}^N f_z(h_z^{-1}(\theta, v_n)) |J_h|}{\int_{\Theta} f_\theta(\theta) \prod_{n=1}^N f_z(h_z^{-1}(\theta, v_n)) |J_h| d\theta} = \frac{f_\theta(\theta) \prod_{n=1}^N f_z(h_z^{-1}(\theta, v_n)) |J_h|}{const}$$

$$\int_{\Theta} f_\theta(\theta) \prod_{n=1}^N f_z(h_z^{-1}(\theta, v_n)) |J_h| d\theta = const \text{ for given sequence } V_N$$

$$f'(\theta|V_N) \propto f_\theta(\theta) \prod_{n=1}^N f_z(h_z^{-1}(\theta, v_n)) |J_h|$$



where:





Bayesian method

- Example

Noise description: $f_z(z) = \frac{1}{\sigma_z \sqrt{2\pi}} \exp\left[-\frac{z^2}{2\sigma_z^2}\right]$

A priori distribution: $f_\theta(\theta) = \frac{1}{\sigma_\theta \sqrt{2\pi}} \exp\left[-\frac{(\theta - m_\theta)^2}{2\sigma_\theta^2}\right]$

Measurement system description: $v = h(\theta, z) = \theta + z, \quad z = h_z^{-1}(\theta, v) = v - \theta$

Loss function: $L(\theta, \bar{\theta}) = -\delta(\theta - \bar{\theta})$



Bayesian method

- Example

Jacobi matrix:
$$J_h = \frac{\partial h_z^{-1}(\theta, v)}{\partial v} = \frac{d}{dv} (v - \theta) = 1$$

Probability density function of the observed value:
$$f_v(v|\theta) = \frac{1}{\sigma_z \sqrt{2\pi}} \exp\left[-\frac{(v - \theta)^2}{2\sigma_z^2}\right] \cdot |1|$$

A posteriori probability density function:

$$\begin{aligned} f'(\theta|V_N) &\propto f_\theta(\bar{\theta}) \prod_{n=1}^N f_z(h_z^{-1}(\theta, v_n)) |J_h| = \\ &= \frac{1}{\sigma_\theta \sqrt{2\pi}} \exp\left[-\frac{(\bar{\theta} - m_\theta)^2}{2\sigma_\theta^2}\right] \prod_{n=1}^N \frac{1}{\sigma_z \sqrt{2\pi}} \exp\left[-\frac{(v_n - \bar{\theta})^2}{2\sigma_z^2}\right] \end{aligned}$$



Bayesian method

- Example

A posteriori probability density function after transformation:

$$f'(\theta|V_N) \propto f_\theta(\bar{\theta}) \prod_{n=1}^N f_z(h_z^{-1}(\bar{\theta}, v_n)) |J_h| =$$

$$= \frac{1}{\sigma_\theta \sqrt{2\pi}} \left(\frac{1}{\sigma_z \sqrt{2\pi}} \right)^N \exp \left[-\frac{(\bar{\theta} - m_\theta)^2}{2\sigma_\theta^2} - \sum_{n=1}^N \frac{(v_n - \bar{\theta})^2}{2\sigma_z^2} \right]$$

For loss function: $L(\theta, \bar{\theta}) = -\delta(\theta - \bar{\theta})$ the conditional risk

$$r(\bar{\theta}, V_N) \stackrel{\text{df}}{=} E_{\bar{\theta}} [L(\theta, \bar{\theta}) | V_N] = \int_{\Theta} L(\theta, \bar{\theta} = \bar{\Psi}(V_N)) f'(\theta | V_N) d\theta = -f'(\theta | V_N) \propto$$

$$\propto -f_\theta(\bar{\theta}) \prod_{n=1}^N f_z(h_z^{-1}(\bar{\theta}, v_n)) |J_h| = -\frac{1}{\sigma_\theta \sqrt{2\pi}} \left(\frac{1}{\sigma_z \sqrt{2\pi}} \right)^N \exp \left[-\frac{(\bar{\theta} - m_\theta)^2}{2\sigma_\theta^2} - \sum_{n=1}^N \frac{(v_n - \bar{\theta})^2}{2\sigma_z^2} \right]$$



Bayesian method

- Example

Estimation algorithm:

$$\theta_N = \Psi_N(V_N) = \frac{m_\theta + \left(\frac{\sigma_\theta}{\sigma_z}\right)^2 \sum_{n=1}^N v_n}{1 + \left(\frac{\sigma_\theta}{\sigma_z}\right)^2 N}$$

Discussion:

1° N – small number

$(\sigma_z \gg \sigma_\theta)$ – poor measurements

2° $N \rightarrow \infty$

$(\sigma_z \ll \sigma_\theta)$ – good measurements

$$\theta_N \approx m_\theta$$

$$\theta_N \approx \frac{1}{N} \sum_{n=1}^N v_n$$



Thank you for attention

