Computer Science

Jerzy Świątek

Systems Modelling and Analysis

Choose yourself and new technologies

L.2. Physical signal characteristic

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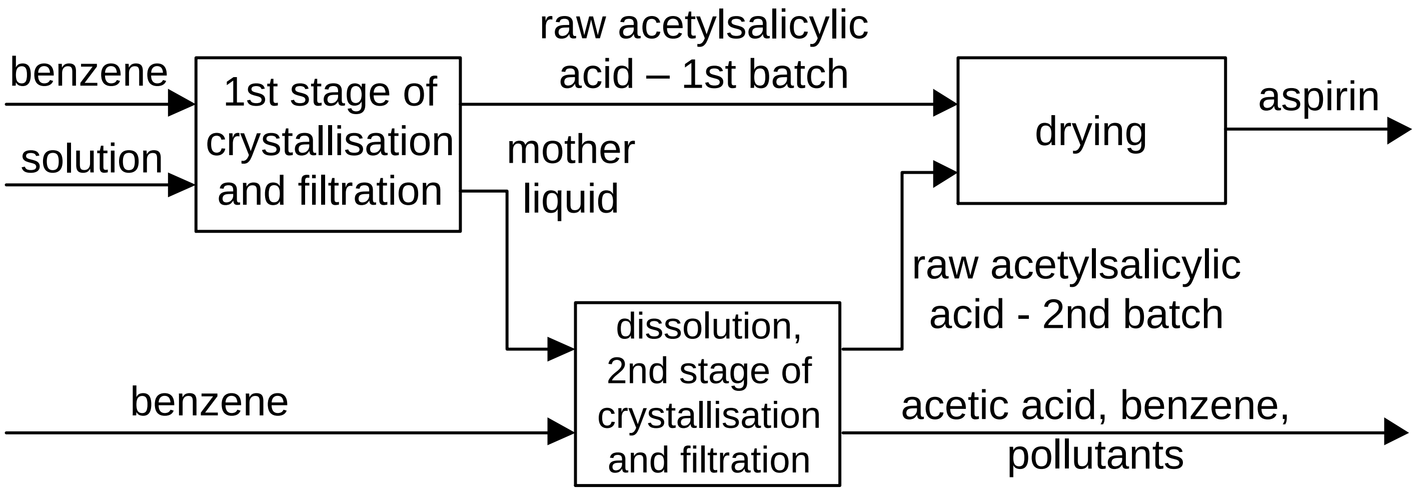
Complex systems description



Complex system of chemical nature

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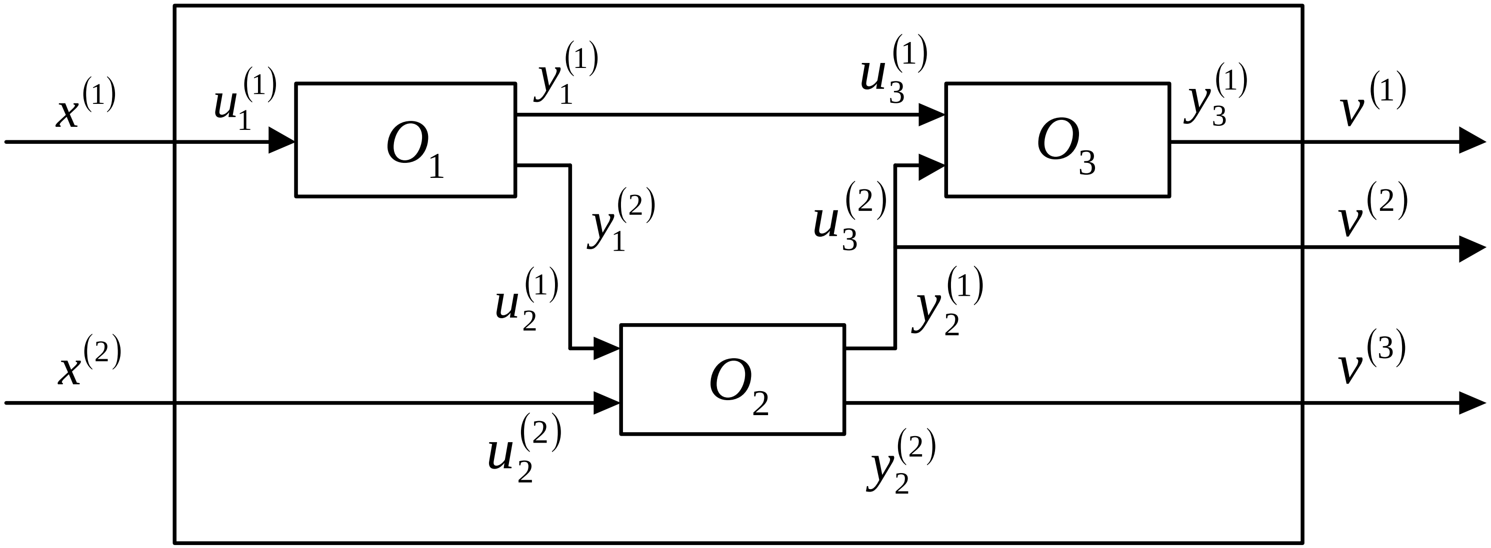
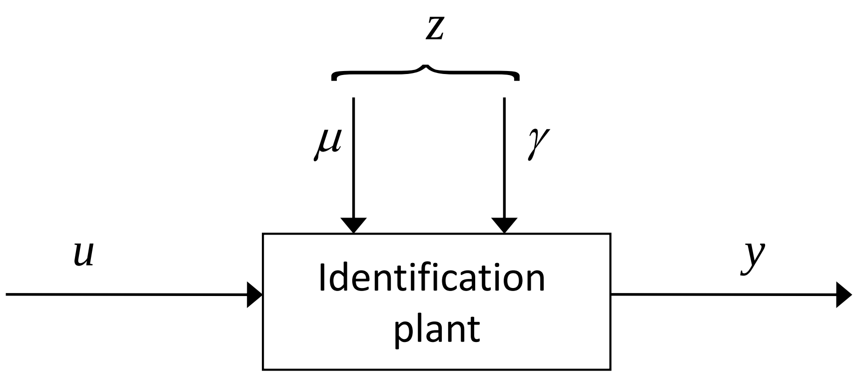
Complex systems description



Example of complex system

Determination of the plant



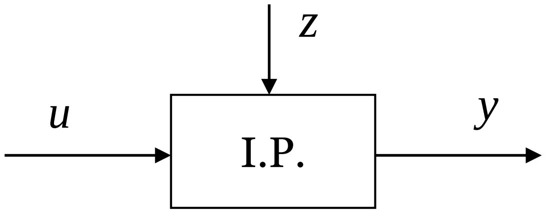
u – input

y – output

μ – measured disturbances

γ – unmeasured disturbances

Static case (1)



u - input $\quad u \in \mathrm{U} \subseteq \mathrm{R}^{S}$

y - output $\quad y \in \mathrm{Y} \subseteq \mathrm{R}^{L}$

z - noise $\quad z \in \mathrm{Z} \subseteq \mathrm{R}^{L}$

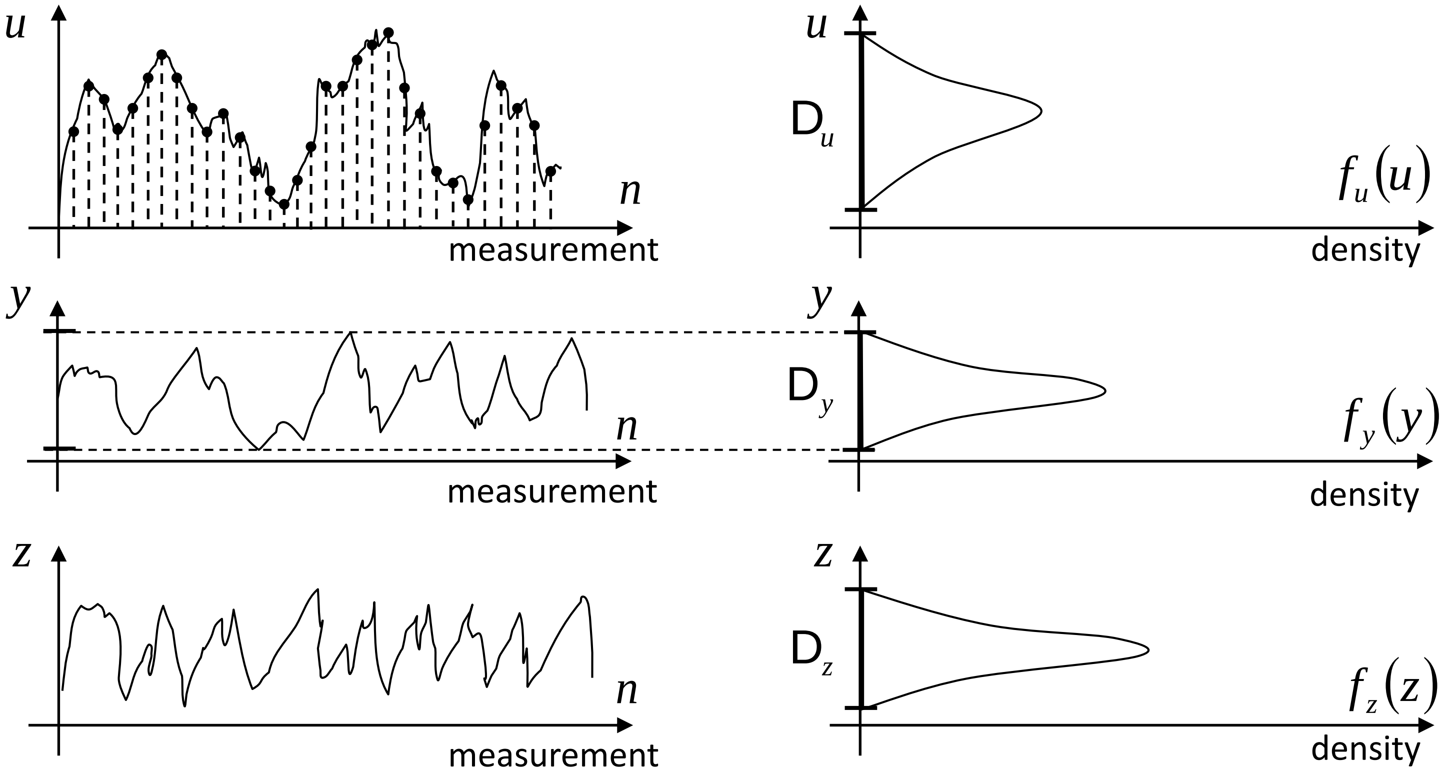
$\mathrm{U} : ? \quad \mathrm{Y} : ? \quad \mathrm{Z} : ?$

$$\mathrm{U} = \left\{ u \in \mathrm{R}^{S},\ \varphi_u(u) = \bar{0}_{L_u},\ \psi_u(u) \le \bar{0}_{M_u} \right\}$$

$$\mathrm{Y} = \left\{ y \in \mathrm{R}^{L},\ \varphi_y(y) = \bar{0}_{L_y},\ \psi_y(u) \le \bar{0}_{M_y} \right\}$$

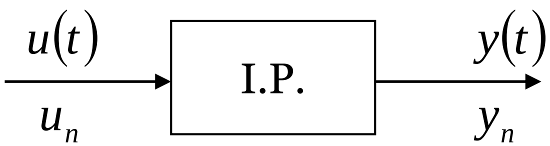
$$\mathrm{Z} = \left\{ z \in \mathrm{R}^{L},\ \varphi_z(z) = \bar{0}_{L_z},\ \psi_z(u) \le \bar{0}_{M_z} \right\}$$

Static case (2)



n – measurement step

Dynamic case



$u(t)$ - input signal (*continuous*)

$y(t)$ - output signal (*continuous*)

u_n - input signal (*discrete*)

y_n - output signal (*discrete*)

$f(t)$ - dynamic signal (*continuous*)

f_n - dynamic signal (*discrete*)

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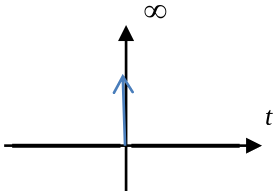
Frequently used continuous signals (1)

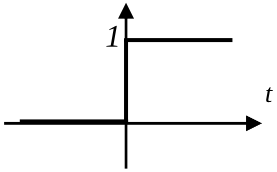
$\delta(t)$ - Dirac delta

$$\delta(t) \overset{\text{df}}{=} \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$



$\mathbf{1}(t)$ - unit step

$$\mathbf{1}(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



Properties:

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\int_{-\infty}^{\infty} f(t-t_0)\delta(t-t_0)dt = f(t_0)$$

$$\frac{d}{dt}\mathbf{1}(t) = \delta(t)$$

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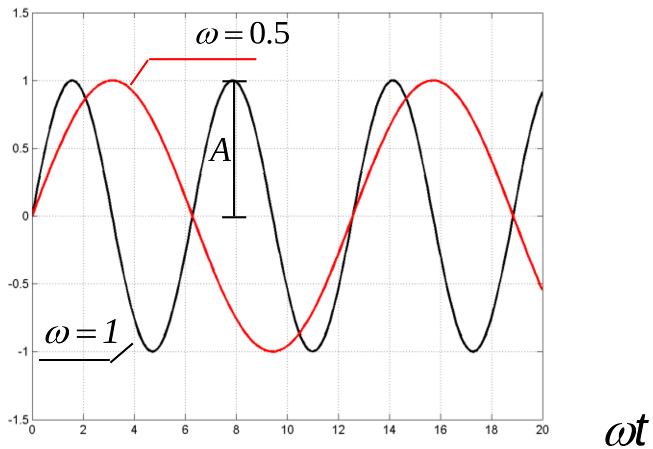
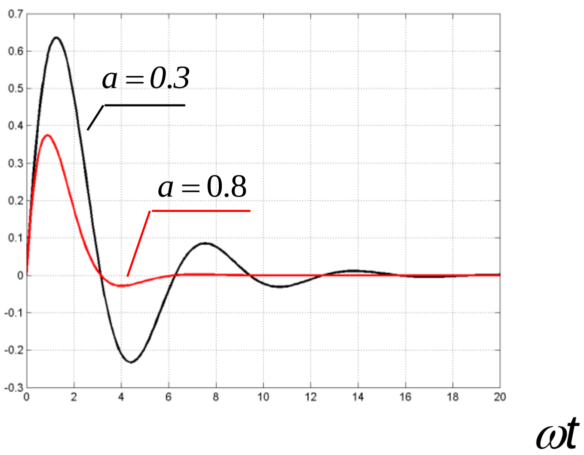
Frequently used continuous signals (2)

$A\sin\omega t$ – sine wave

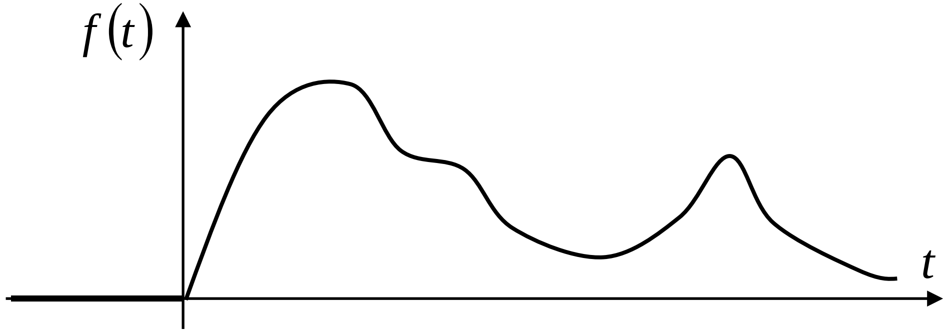


$e^{-at}\sin\omega t$ – damped sine wave



Laplace Transform (1)

Let us consider a function $f(t)$, for which $f(t)=0$ for $t<0$



Laplace transform definition:

$$F(s)=\mathcal{L}\left[f(t)\right]\overset{\text{df}}{=}\int_{o}^{\infty} f(t)\cdot e^{-st}dt$$

where:

s - complex variable: $s=\sigma+j\omega$

Laplace Transform (2)
Definition

Inverse Laplace transform:

$$f(t) = \mathcal{L}^{-1}[F(s)] \overset{\text{df}}{=} \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)\cdot e^{st} ds$$

$$\begin{array}{ccc} f(t) & \rightarrow & F(s) \\ & tranformations & \downarrow \\ f'(t) & \leftarrow & F'(s) \end{array}$$

Laplace Transform (15)
Frequently used continuous signals

$f(t)$	$F(s)$
$\delta(t)$	1
$\mathbf{1}(t)$	$\frac{1}{s}$
$\mathbf{1}(t)\,t$	$\frac{1}{s^2}$
$\mathbf{1}(t)\frac{t^{k-1}}{(k-1)!}$	$\frac{1}{s^k}$
$\mathbf{1}(t)\,e^{-at}$	$\frac{1}{s+a}$
$\mathbf{1}(t)\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
$\mathbf{1}(t)\cos\omega t$	$\frac{s}{s^2+\omega^2}$

Laplace Transform (3)
Properties

1. Linearity:

$$\mathcal{L}\left[a_1 f_1(t)+a_2 f_2(t)\right]=a_1\mathcal{L}\left[f_1(t)\right]+a_2\mathcal{L}\left[f_2(t)\right]=a_1F_1(s)+a_2F_2(s)$$

Proof:

$$\mathcal{L}\left[a_1 f_1(t)+a_2 f_2(t)\right]=\int_0^\infty \left(a_1 f_1(t)+a_2 f_2(t)\right)e^{-st}dt=\int_0^\infty \left(a_1 f_1(t)e^{-st}+a_2 f_2(t)e^{-st}\right)dt=$$

$$=\int_0^\infty a_1 f_1(t)e^{-st}dt+\int_0^\infty a_2 f_2(t)e^{-st}dt=a_1\int_0^\infty f_1(t)e^{-st}dt+a_2\int_0^\infty f_2(t)e^{-st}dt=a_1\mathcal{L}\left[f_1(t)\right]+a_2\mathcal{L}\left[f_2(t)\right]$$

Laplace Transform (4)
Properties

2. Differentiation:

a) the first derivative $\mathcal{L}\left[f'(t)\right]=s\mathcal{L}\left[f(t)\right]-f(0)$

Proof: $\mathcal{L}\left[f'(t)\right]=\int_0^\infty f'(t)e^{-st}dt$

By making use of integration by parts, where $h(t)=e^{-st}$, $h'(t)=-se^{-st}$, we have:

$$\int_0^\infty f'(t)e^{-st}dt=\left[f(t)h(t)\right]_0^\infty-\int_0^\infty f(t)h'(t)dt=\left[f(t)e^{-st}\right]_0^\infty-\int_0^\infty f(t)\left(-se^{-st}\right)dt$$

Assuming $\mathrm{Re}(s)>0$ we obtain:

$$\lim_{t\to\infty} f(t)e^{-st}-f(0)e^{-s\cdot 0}+s\int_0^\infty f(t)e^{-st}\,dt=s\mathcal{L}\left[f(t)\right]-f(0)$$

Laplace Transform (5)
Properties

2. Differentiation:

b) the second derivative $\mathcal{L}\left[f''(t)\right]=s^2\mathcal{L}\left[f(t)\right]-sf(0)-f'(0)$

Proof: $\mathcal{L}\left[f''(t)\right]=s\mathcal{L}\left[f'(t)\right]-f'(0)=s\left(s\mathcal{L}\left[f(t)\right]-f(0)\right)-f'(0)=$

$=s^2\mathcal{L}\left[f(t)\right]-sf(0)-f'(0)$

Laplace Transform (6)
Properties

2. Differentiation:

c) the n - th order derivative

$$\mathcal{L}\left[f^{(n)}(t)\right]=s^n\mathcal{L}\left[f(t)\right]-s^{n-1}f(0)-s^{n-2}f'(0)-\ldots-f^{(n-1)}(0)$$

Proof:

$$\mathcal{L}\left[f^{(n)}(t)\right]=s\mathcal{L}\left[f^{(n-1)}(t)\right]-f^{(n-1)}(0)$$

$$\mathcal{L}\left[f^{(n)}(t)\right]=s^2\mathcal{L}\left[f^{(n-2)}(t)\right]-f^{(n-1)}(0)-sf^{(n-2)}(0)$$

$$\mathcal{L}\left[f^{(n)}(t)\right]=s^3\mathcal{L}\left[f^{(n-3)}(t)\right]-f^{(n-1)}(0)-sf^{(n-2)}(0)-s^2f^{(n-3)}(0)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathcal{L}\left[f^{(n)}(t)\right]=s^n\mathcal{L}\left[f(t)\right]-f^{(n-1)}(0)-sf^{(n-2)}(0)-\ldots-s^{n-2}f'(0)-s^{n-1}f(0)$$

Laplace Transform (7)
Properties

3. Integration: $$\mathcal{L}\left[\int_o^t f(\tau)d\tau\right] = \frac{1}{s}\mathcal{L}\left[f(t)\right]$$

Proof: Let $h(t) = \int_o^t f(\tau)d\tau$. Note that $h(0)=0$ and $h'(t) = f(t)$. Then:

$\mathcal{L}\left[h'(t)\right] = s\mathcal{L}\left[h(t)\right] - h(0) = s\mathcal{L}\left[h(t)\right]$ which gives:

$\mathcal{L}\left[h(t)\right] = \frac{1}{s}\mathcal{L}\left[h'(t)\right]$ and after substitution:

$$\mathcal{L}\left[\int_o^t f(\tau)d\tau\right] = \frac{1}{s}\mathcal{L}\left[f(t)\right]$$

Laplace Transform (8)
Properties

4. Multiplication of a function by t : $\mathcal{L}\left[tf(t)\right] = -\frac{d}{ds}\mathcal{L}\left[f(t)\right]$

Proof:

$$-\frac{d}{ds}\mathcal{L}\left[f(t)\right] = -\frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = -\int_0^\infty \frac{d}{ds}e^{-st}f(t)dt = \int_0^\infty te^{-st}f(t)dt =$$

$$= \int_0^\infty tf(t)e^{-st}dt = \mathcal{L}\left[tf(t)\right]$$

Laplace Transform (9)
Properties

5. Division of a function by t : $\mathcal{L}\left[\frac{f(t)}{t}\right]=\int_s^\infty \mathcal{L}\left[f(t)\right]ds$

Proof:

$$\int_s^\infty \mathcal{L}\left[f(t)\right]ds=\int_s^\infty\left(\int_0^\infty f(t)e^{-st}dt\right)ds=\int_0^\infty f(t)\left(\int_s^\infty e^{-st}ds\right)dt=\int_0^\infty f(t)\left(-\frac{1}{t}\left[e^{-st}\right]_s^\infty\right)dt=$$

$$=\int_0^\infty f(t)\left(0+\frac{1}{t}e^{-st}\right)dt=\int_0^\infty \frac{f(t)}{t}e^{-st}dt=\mathcal{L}\left[\frac{f(t)}{t}\right]$$

Laplace Transform (10)
Properties

6. Multiplication of a function by e^{at}: $\mathcal{L}\left[e^{at} f(t)\right] = F(s-a)$

Proof:

$$\mathcal{L}\left[e^{at} f(t)\right] = \int_0^\infty e^{at} f(t) e^{-st} dt = \int_0^\infty e^{at-st} f(t) dt = \int_0^\infty e^{-(s-a)t} f(t) dt = F(s-a)$$

Laplace Transform (11)
Properties

7. Change of time scale: $\mathcal{L}\left[f(at)\right] = \frac{1}{a}F\left(\frac{s}{a}\right),$ where $a > 0$

Proof:

$$\mathcal{L}\left[f(at)\right] = \int_0^\infty f(at)e^{-st}dt$$

Taking $x = at$ we obtain:

$$\int_0^\infty f(x)e^{-s\frac{x}{a}}d\frac{x}{a} = \frac{1}{a}\int_0^\infty f(x)e^{-\frac{s}{a}x}dx = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Laplace Transform (12)
Properties

8. Delay: $\mathcal{L}\left[f(t-t_0)\cdot 1(t-t_0)\right] = e^{-st_0}\mathcal{L}\left[f(t)\right]$

Proof:

$$\mathcal{L}\left[f(t-t_0)\cdot 1(t-t_0)\right] = \int_0^{\infty} f(t-t_0)\cdot 1(t-t_0)e^{-st}dt = \int_{t_0}^{\infty} f(t-t_0)e^{-st}dt$$

Taking $\tau = t - t_0$ we obtain:

$$\int_0^{\infty} f(\tau)e^{-s(\tau+t_0)}dt = \int_0^{\infty} f(\tau)e^{-s\tau}e^{-st_0}d\tau = e^{-st_0}\int_0^{\infty} f(\tau)e^{-s\tau}d\tau = e^{-st_0}\mathcal{L}\left[f(t)\right]$$

Laplace Transform (13)
Properties

9. Convolution: $$f_1(t) * f_2(t) \overset{\text{df}}{=} \int_0^t f_1(t-\tau) f_2(\tau) d\tau$$

$$\mathcal{L}\left[\int_0^t f_1(t-\tau) f_2(\tau) d\tau\right] = \mathcal{L}\left[f_1(t)\right]\mathcal{L}\left[f_2(t)\right]$$

Proof:

Note that $f_1(t-\tau)\mathbf{1}(t-\tau) = 0$ for $\tau > t$: $$\int_0^t f_1(t-\tau) f_2(\tau) d\tau = \int_0^\infty f_1(t-\tau)\mathbf{1}(t-\tau) f_2(\tau) d\tau$$

Then:

$$\mathcal{L}\left[\int_0^t f_1(t-\tau) f_2(\tau) d\tau\right] = \mathcal{L}\left[\int_0^\infty f_1(t-\tau)\mathbf{1}(t-\tau) f_2(\tau) d\tau\right] = \int_0^\infty e^{-st}\left[\int_0^\infty f_1(t-\tau)\mathbf{1}(t-\tau) f_2(\tau) d\tau\right] dt$$

Laplace Transform (14)
Properties

Proof:

Substituting $t-\tau=\lambda$ and changing the order od integration gives:

$$\mathcal{L}\left[\int_0^t f_1(t-\tau)\,f_2(\tau)d\tau\right]=\int_0^\infty f_1(t-\tau)\mathbf{1}(t-\tau)e^{-st}dt\int_0^\infty f_2(\tau)d\tau=$$

$$=\int_0^\infty f_1(\lambda)e^{-s(\lambda+\tau)}d\lambda\int_0^\infty f_2(\tau)d\tau=\int_0^\infty f_1(\lambda)e^{-s\lambda}d\lambda\int_0^\infty f_2(\tau)e^{-s\tau}d\tau=$$

$$=\mathcal{L}\left[f_1(t)\right]\mathcal{L}\left[f_2(t)\right]$$

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Transformata Laplace'a

$$F(s)=\frac{s^{\nu}b_{\nu}+s^{\nu-1}b_{\nu-1}+\cdots+sb_1+b_0}{s^m+a_{m-1}s^{m-1}+\cdots+a_1s+a_0},\quad m\geq\nu$$

s_i - *i*-th roots i=1,2, …, m
m - number of roots

$$F(s)=\frac{s^{\nu}b_{\nu}+s^{\nu-1}b_{\nu-1}+\cdots+sb_1+b_0}{(s-s_1)(s-s_2)\cdots(s-s_m)},\quad m\geq\nu$$

$$F(s)=\sum_{i=1}^{m}\frac{A_i}{(s-s_i)}=\frac{A_1}{(s-s_1)}+\frac{A_2}{(s-s_2)}+\cdots+\frac{A_m}{(s-s_m)}$$

$$f(t)=1(t)\sum_{i=1}^{m}A_ie^{s_it}=1(t)A_1e^{s_1t}+1(t)A_2e^{s_2t}+\cdots+1(t)A_me^{s_mt}$$

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Laplace Transform (15)

Properties

$$F(s)=\frac{b_\nu+b_{\nu-1}+\cdots+b_1+b_0}{s^m+a_{m-1}s^{m-1}+\cdots+a_1 s+a_0},\quad m\geq\nu$$

$$F(s)=\sum_{i=1}^{r}\sum_{j=1}^{\alpha_i}\frac{A_{ij}}{(s-s_i)^j}\qquad f(t)=1(t)\sum_{i=1}^{r}\sum_{j=1}^{\alpha_i}\frac{A_{ij}}{(j-1)!}t^{j-1}e^{s_i t}\qquad \sum_{i=1}^{r}\alpha_i=m$$

s_i - roots α_i - number of *i- th* roots *r* – number of different roots

$$\mathcal{L}^{-1}\left[\frac{1}{(s-s_i)^j}\right]=1(t)\frac{t^{j-1}}{(j-1)!}e^{s_i t}$$

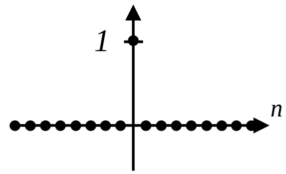
Laplace Transform (15)
Frequently used continuous signals

$f(t)$	$F(s)$
$\delta(t)$	1
$\mathbf{1}(t)$	$\frac{1}{s}$
$\mathbf{1}(t)\,t$	$\frac{1}{s^2}$
$\mathbf{1}(t)\frac{t^{k-1}}{(k-1)!}$	$\frac{1}{s^k}$
$\mathbf{1}(t)\,e^{-at}$	$\frac{1}{s+a}$
$\mathbf{1}(t)\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
$\mathbf{1}(t)\cos\omega t$	$\frac{s}{s^2+\omega^2}$

Frequently used discrete signals (1)

δ_n - Kronecker delta

$$\delta_n = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

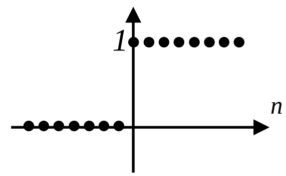


Properties:

$$\sum_{n=-\infty}^{\infty} \delta_{n-k} a_{n-k} = a_k$$

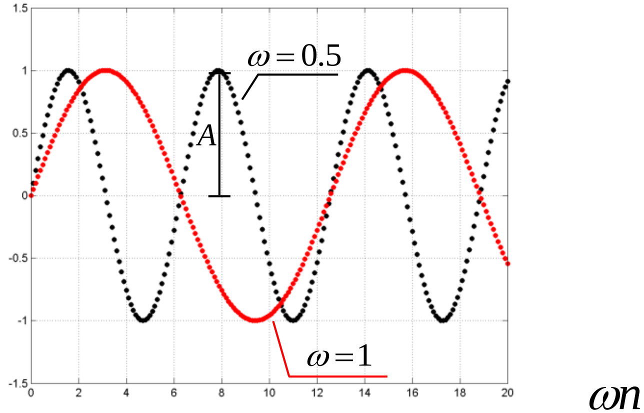
$\mathbf{1}_n$ - unit step

$$\mathbf{1}_n = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



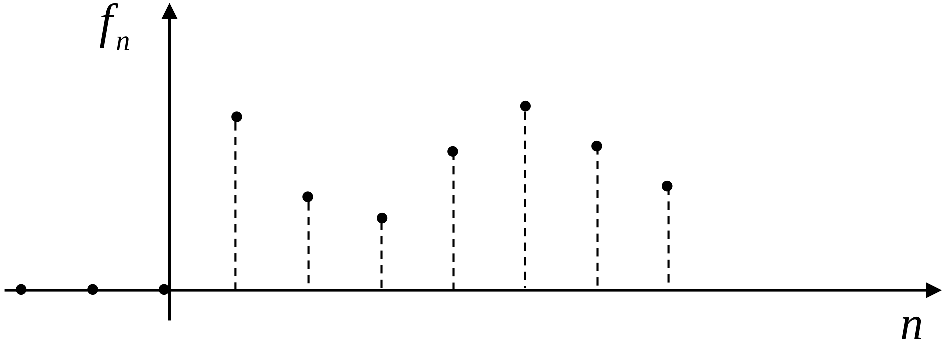
Frequently used discrete signals (2)

$A\sin\omega n$ - discrete sine wave



Z – Transform (1)

Let us consider a sequence of numbers f_n for which $f_n = 0$ for $n < 0$



For each n, $|f_n| < \infty$ holds $\lim_{n \to \infty} |f_n z^{-n}| = 0$

z - complex variable: $z = \sigma + j\omega$

Z – Transform (2)
Definition

Transform definition: $$F(z)=\mathrm{Z}\left[f_n\right]\overset{\mathrm{df}}{=}\sum_{n=0}^{\infty} f_n z^{-n}$$

Inverse transform: $$f_n=\mathrm{Z}^{-1}\left[F(z)\right]=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(z)z^{n-1}dz$$

Z - Transform

$$
\begin{array}{ccc}
f_n & \rightarrow & F(z) \\
transformations & & \downarrow \\
f_n' & \leftarrow & F'(z)
\end{array}
$$

Z – Transform (12)
Frequently used discrete signals

f_n	$F(z)$
δ_n	1
$\mathbf{1}_n$	$\frac{z}{z-1}$
$\mathbf{1}_n \cdot n$	$\frac{z}{(z-1)^2}$
$\mathbf{1}_n \cdot n^2$	$\frac{z^2+z}{(z-1)^2}$
$\mathbf{1}_n \cdot a^n$	$\frac{z}{z-a}$
$\mathbf{1}_n \cdot \sin \omega n$	$\frac{z \sin \omega}{z^2 - 2z\cos\omega + 1}$
$\mathbf{1}_n \cdot \cos \omega n$	$\frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}$

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Z – Transform (3)
Properties

1. Linearity: $\mathrm{Z}\left[af_n + bg_n\right] = a\mathrm{Z}\left[f_n\right] + b\mathrm{Z}\left[g_n\right]$

Proof:

$$\mathrm{Z}\left[af_n + bg_n\right] = \sum_{n=0}^{\infty}\left(af_n + bg_n\right)z^{-n} = \sum_{n=0}^{\infty}\left(af_n z^{-n} + bg_n z^{-n}\right) =$$

$$= \sum_{n=0}^{\infty} af_n z^{-n} + \sum_{n=0}^{\infty} bg_n z^{-n} = a\sum_{n=0}^{\infty} f_n z^{-n} + b\sum_{n=0}^{\infty} g_n z^{-n} = a\mathrm{Z}\left[f_n\right] + b\mathrm{Z}\left[g_n\right]$$

Z – Transform (4)
Properties

2. Shifting forward : $$\mathrm{Z}\left[f_{n+m}\right]=z^{m}\left[\mathrm{Z}\left[f_{n}\right]-\sum_{k=0}^{m-1} f_{k} z^{-k}\right]$$

Proof:

For $m=1$: $$\mathrm{Z}\left[f_{n+1}\right]=\sum_{n=0}^{\infty} f_{n+1} z^{-n}$$

Let $j=n+1$:

$$\sum_{n=0}^{\infty} f_{n+1} z^{-n}=\sum_{j=1}^{\infty} f_{j} z^{-j+1}=\sum_{j=1}^{\infty} f_{j} z^{-j} z=z \sum_{j=1}^{\infty} f_{j} z^{-j}=$$

$$=z\left[\sum_{j=0}^{\infty} f_{j} z^{-j}-f_{0} z^{0}\right]=z\left[\mathrm{Z}\left[f_{n}\right]-f_{0}\right]$$

Z – Transform (5)

Properties

2. Shifting forward : $\text{Z}\left[f_{n+m}\right]=z^{m}\left[\text{Z}\left[f_{n}\right]-\sum_{k=0}^{m-1} f_{k} z^{-k}\right]$

Proof:

For $m=2$: $\text{Z}\left[f_{n+2}\right]=\sum_{n=0}^{\infty} f_{n+2} z^{-n}$

Let $j=n+2$:

$$\sum_{n=0}^{\infty} f_{n+2} z^{-n}=\sum_{j=2}^{\infty} f_{j} z^{-j+2}=\sum_{j=2}^{\infty} f_{j} z^{-j} z^{2}=z^{2} \sum_{j=2}^{\infty} f_{j} z^{-j}=$$

$$=z^{2}\left[\sum_{j=0}^{\infty} f_{j} z^{-j}-f_{0} z^{0}-f_{1} z^{-1}\right]=z^{2}\left[\text{Z}\left[f_{n}\right]-f_{0}-f_{1} z^{-1}\right]$$

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Z – Transform (6)
Properties

2. Shifting forward : $$\mathrm{Z}\left[f_{n+m}\right]=z^m\left[\mathrm{Z}\left[f_n\right]-\sum_{k=0}^{m-1} f_k z^{-k}\right]$$

Proof:

$$\mathrm{Z}\left[f_{n+1}\right]=z\left[\mathrm{Z}\left[f_n\right]-f_0\right]$$
$$\mathrm{Z}\left[f_{n+2}\right]=z^2\left[\mathrm{Z}\left[f_n\right]-f_0-f_1 z^{-1}\right]$$
$$\mathrm{Z}\left[f_{n+3}\right]=z^3\left[\mathrm{Z}\left[f_n\right]-f_0-f_1 z^{-1}-f_2 z^{-2}\right]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mathrm{Z}\left[f_{n+m}\right]=z^m\left[\mathrm{Z}\left[f_n\right]-f_0-f_1 z^{-1}-f_2 z^{-2}-\ldots-f_{m-1} z^{-(m-1)}\right]=$$
$$=z^m\left[\mathrm{Z}\left[f_n\right]-\sum_{k=0}^{m-1} f_k z^{-k}\right]$$

Z – Transform (7)
Properties

3. Shifting backward: $\mathrm{Z}\left[f_{n-m}\right]=z^{-m}\mathrm{Z}\left[f_{n}\right]$

Proof:

For $m=1$: $\mathrm{Z}\left[f_{n-1}\right]=\sum_{n=0}^{\infty} f_{n-1}z^{-n}$

Let $j=n-1$:

$$\sum_{n=0}^{\infty} f_{n-1}z^{-n}=\sum_{j=-1}^{\infty} f_{j}z^{-j-1}=\sum_{j=-1}^{\infty} f_{j}z^{-j}z^{-1}=z^{-1}\sum_{j=-1}^{\infty} f_{j}z^{-j}$$

Since $f_j=0$ for $j<0$, we have: $z^{-1}\sum_{j=-1}^{\infty} f_{j}z^{-j}=z^{-1}\sum_{j=0}^{\infty} f_{j}z^{-j}=z^{-1}\mathrm{Z}\left[f_{n}\right]$

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Z – Transform (8)
Properties

3. Shifting backward: $\mathrm{Z}\left[f_{n-m}\right]=z^{-m}\mathrm{Z}\left[f_n\right]$

Proof:

For $m=2$: $\mathrm{Z}\left[f_{n-2}\right]=\sum_{n=0}^{\infty} f_{n-2} z^{-n}$

Let $j=n-2$:

$$\sum_{n=0}^{\infty} f_{n-2} z^{-n}=\sum_{j=-2}^{\infty} f_j z^{-j-2}=\sum_{j=-2}^{\infty} f_j z^{-j} z^{-2}=z^{-2} \sum_{j=-2}^{\infty} f_j z^{-j}$$

Since $f_j=0$ for $j<0$, we have: $z^{-2} \sum_{j=-2}^{\infty} f_j z^{-j}=z^{-2} \sum_{j=0}^{\infty} f_j z^{-j}=z^{-2}\mathrm{Z}\left[f_n\right]$

Z – Transform (9)
Properties

3. Shifting backward: $\mathrm{Z}\left[f_{n-m}\right]=z^{-m}\mathrm{Z}\left[f_n\right]$

Proof:

$$\mathrm{Z}\left[f_{n-1}\right]=z^{-1}\mathrm{Z}\left[f_n\right]$$
$$\mathrm{Z}\left[f_{n-2}\right]=z^{-2}\mathrm{Z}\left[f_n\right]$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mathrm{Z}\left[f_{n-m}\right]=z^{-m}\mathrm{Z}\left[f_n\right]$$

Z – Transform (10)
Properties

4. Scaling: $\mathcal{Z}\left[nf_n\right] = -z\frac{d}{dz}\mathcal{Z}\left[f_n\right]$

Proof:

$$\mathcal{Z}\left[nf_n\right] = \sum_{n=0}^{\infty} nf_n z^{-n} = z\sum_{n=0}^{\infty} nf_n z^{-n-1} = -z\sum_{n=0}^{\infty} f_n\left(-nz^{-n-1}\right) =$$

$$= -z\sum_{n=0}^{\infty} f_n \frac{d}{dz}\left(z^{-n}\right) = -z\frac{d}{dz}\sum_{n=0}^{\infty} f_n z^{-n} = -z\frac{d}{dz}\mathcal{Z}\left[f_n\right]$$

Z – Transform (11)
Properties

5. Convolution:

$$f_n * g_n \overset{\text{df}}{=} \sum_{k=0}^{n} f_k g_{n-k}$$

$$Z\left[\sum_{k=0}^{n} f_k g_{n-k}\right] = Z\,[f_n]\,Z\,[g_n]$$

Proof:

$$Z\left[\sum_{k=0}^{n} f_k g_{n-k}\right] = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} f_k g_{n-k}\right) z^{-n} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} f_k g_{n-k}\right) z^{-n} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} f_k g_{n-k} z^{-n} = \sum_{k=0}^{\infty} f_k \sum_{n=0}^{\infty} g_{n-k} z^{-n}$$

Let $r = n - k$:

$$\sum_{k=0}^{\infty} f_k \sum_{n=0}^{\infty} g_{n-k} z^{-n} = \sum_{k=0}^{\infty} f_k \sum_{r=0}^{\infty} g_r z^{-r-k} = \sum_{k=0}^{\infty} f_k z^{-k} \sum_{r=0}^{\infty} g_r z^{-r} = Z\,[f_n]\,Z\,[g_n]$$

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Transformata Laplace'a

$$F(z)=\frac{z^{\nu}b_{\nu}+z^{\nu-1}b_{\nu-1}+\cdots+zb_1+b_0}{z^m+a_{m-1}z^{m-1}+\cdots+a_1z+a_0},\quad m\geq\nu$$

s_i - *i*-th roots i=1,2, …, m
m - number of roots

$$F(z)=\frac{z^{\nu}b_{\nu}+z^{\nu-1}b_{\nu-1}+\cdots+zb_1+b_0}{(z-s_1)(z-s_2)\cdots(z-s_m)},\quad m\geq\nu$$

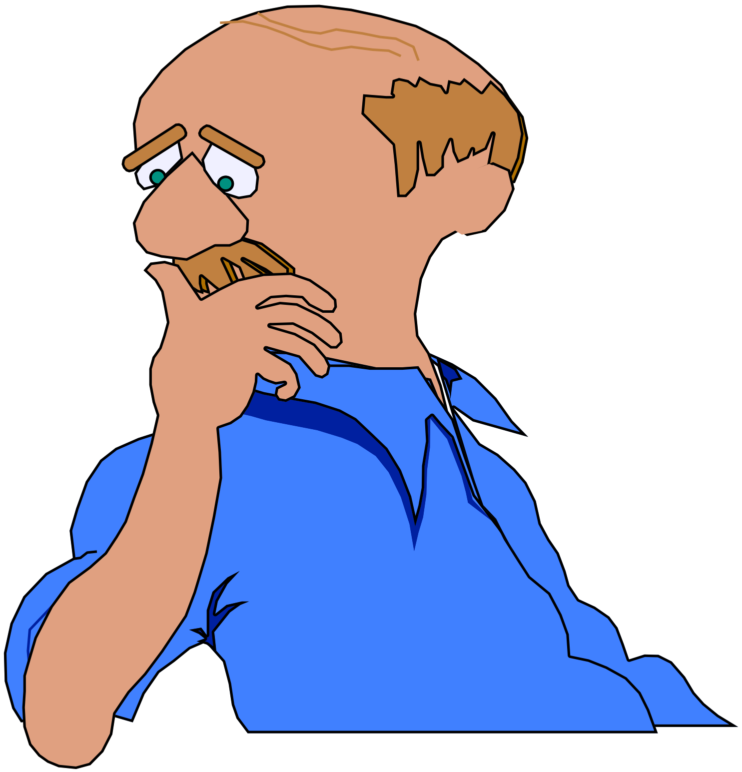
$$F(z)=\sum_{i=1}^{m}\frac{A_iz}{(z-s_i)}=\frac{A_1z}{(z-s_1)}+\frac{A_2z}{(z-s_2)}+\cdots+\frac{A_mz}{(z-s_m)}$$

$$f_n=\sum_{i=1}^{m}1_nA_is_i^n=1_nA_1s_1^n+1_nA_2s_2^n+\cdots+1_ns_m^nA_m$$

Z – Transform (12)
Frequently used discrete signals

f_n	$F(z)$
δ_n	1
$\mathbf{1}_n$	$\frac{z}{z-1}$
$\mathbf{1}_n \cdot n$	$\frac{z}{(z-1)^2}$
$\mathbf{1}_n \cdot n^2$	$\frac{z^2+z}{(z-1)^2}$
$\mathbf{1}_n \cdot a^n$	$\frac{z}{z-a}$
$\mathbf{1}_n \cdot \sin \omega n$	$\frac{z \sin \omega}{z^2 - 2z\cos\omega + 1}$
$\mathbf{1}_n \cdot \cos \omega n$	$\frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}$

Thank you for attention



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