Computer Science

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Systems Modelling and Analysis

Choose yourself and new technologies

L.19b. Multistage decision making

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MULTISTAGE DECISION MAKING

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$$x_1^*, x_2^*, \ldots, x_S^* \rightarrow F(x_1^*, x_2^*, \ldots, x_S^*) = \min_{x_1, x_2, \ldots, x_S \in \mathrm{D}_x} F(x_1, x_2, \ldots, x_S)$$

$$\mathrm{D}_x = (x_1, x_2, \cdots, x_S)$$

$$\left\{ [x_1\ x_2\ \cdots\ x_S]^T \in \mathrm{R}^S : \varphi_l(x_1, x_2, \cdots, x_S) = 0, l = 1, 2, \ldots, L, \quad \psi_m(x_1, x_2, \cdots, x_S) \le 0, m = 1, 2, \ldots, M \right\}$$

The above task may be solved step by step, selecting a single decision variable to be optimized and relating with remaining decision variables.
Let us denote:

$$F \equiv F_S, \quad \varphi_l \equiv \varphi_{lS}, l = 1, 2, \ldots, L, \quad \psi_m = \psi_{mS}, m = 1, 2, \ldots, M \quad \mathrm{D}_x \equiv \mathrm{D}_{xS}$$

Multistage optimization

Step 1. $x_S^* = G_S(x_1,\ldots,x_{S-1}) \rightarrow F_S(x_1, x_2,\ldots,x_S^*) = \min\limits_{x_S \in D_{xS}} F_S(x_1, x_2,\ldots,x_S)$

The value of the goal function in the optimal solution:

$$F_{S-1}(x_1, x_2,\ldots,x_{S-1}) \overset{\Delta}{=} F_S(x_1, x_2,\ldots,x_S^*) = F_S(x_1, x_2,\ldots,G_S(x_1,\ldots,x_{S-1}))$$

Constraints in the optimal solution:

$$D_{xS-1}(x_1,\ldots,x_{S-1}) \overset{\Delta}{=} D_{xS}(x_1,\ldots,x_{S-1}, x_S^* = G_S(x_1,\ldots,x_{S-1})) =$$

$$\left\{\begin{array}{l} [x_1\ x_2\ \cdots\ x_{S-1}]^T \in R^{S-1}: \\ \varphi_{lS}(x_1, x_2, \cdots, G_S(x_1,\ldots,x_{S-1})) = \varphi_{lS-1}(x_1, x_2, \cdots, x_{S-1}) = 0,\ l = 1, 2, \ldots, L, \\ \psi_{mS}(x_1, x_2, \cdots, G_S(x_1,\ldots,x_{S-1})) = \psi_{mS-1}(x_1, x_2, \cdots, x_{S-1}) \le 0,\ m = 1, 2, \ldots, M \end{array}\right\}$$

Multistage optimization

Step 2. $x^*_{S-1} = G_{S-1}(x_1,\ldots,x_{S-2}) \to F_{S-1}(x_1, x_2,\ldots,x^*_{S-1}) = \min\limits_{x_{S-1}\in D_{xS-1}} F_{S-1}(x_1, x_2,\ldots,x_{S-1})$

The value of the goal function in the optimal solution:

$$F_{S-2}(x_1, x_2,\ldots,x_{S-2}) \stackrel{\Delta}{=} F_{S-1}(x_1, x_2,\ldots,x^*_{S-1}) = F_{S-1}(x_1, x_2,\ldots,G_{S-1}(x_1,\ldots,x_{S-2}))$$

Constraints in the optimal solution:

$$D_{xS-2}(x_1,\ldots,x_{S-2}) \stackrel{\Delta}{=} D_{xS-1}(x_1,\ldots,x_{S-2}, x^*_{S-1} = G_{S-1}(x_1,\ldots,x_{S-2})) =$$
$$\left\{\begin{array}{l} [x_1\ x_2\ \cdots\ x_{S-2}]^T \in R^{S-2}: \\ \varphi_{lS-1}(x_1, x_2, \cdots, G_{S-1}(x_1,\ldots,x_{S-2})) = \varphi_{lS-2}(x_1, x_2, \cdots, x_{S-2}) = 0,\ l = 1, 2, \ldots, L, \\ \psi_{mS-1}(x_1, x_2, \cdots, G_{S-1}(x_1,\ldots,x_{S-2})) = \psi_{mS-2}(x_1, x_2, \cdots, x_{S-2}) \le 0,\ m = 1, 2, \ldots, M \end{array}\right\}$$

Multistage optimization

Step S-1. $$x_2^* = G_2(x_1) \rightarrow F_2(x_1, x_2^*) = \min_{x_2 \in \mathrm{D}_{x2}} F_2(x_1, x_2)$$

The value of the goal function in the optimal solution:

$$F_1(x_1) \stackrel{\Delta}{=} F_2(x_1\, x_2^*) = F_2(x_1, G_2(x_1))$$

Constraints in the optimal solution:

$$\mathrm{D}_{x1}(x_1) \stackrel{\Delta}{=} \mathrm{D}_{xS-1}(x_1, x_2^* = G_1(x_1)) = \left\{ \begin{array}{l} x_1 \in \mathrm{R} : \\ \varphi_{l2}(x_1, G_1(x_1)) = \varphi_{l1}(x_1) = 0, l = 1, 2, \ldots, L, \\ \psi_{m2}(x_1, G_{S-1}(x_1)) = \psi_{m1}(x_1) \le 0, m = 1, 2, \ldots, M \end{array} \right\}$$

Multistage optimization

Step S-1. $x_1^* \to F_1\left(x_1^*\right) = \min\limits_{x_1 \in D_{x1}} F_1\left(x_1\right)$

We may now return to expressions „*G*" determined in the previous steps

$$x_1^*$$

$$x_2^* = G_2\left(x_1^*\right)$$

$$\vdots$$

$$x_{S-1}^* = G_{S-1}\left(x_1^*, x_2^* \ldots, x_{S-1}^*\right)$$

$$x_S^* = G_S\left(x_1^*, x^*, \ldots, x_{S-1}^*\right)$$

Multistage optimization

Example: $$F(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2 = F_2(x_1, x_2)$$

Step 1. $$x_2^* = G_2(x_1) \rightarrow F_2(x_1, x_2^*) = \min_{x_2} \left\{ x_1^2 + x_1 x_2 + x_2^2 \right\}$$

$$\frac{\partial}{\partial x_2} F_2(x_1, x_2^*) = 2x_1 + x_2^* = 0 \Rightarrow x_2^* = -\frac{1}{2} x_1 = G_2(x_1)$$

$$F_1(x_1) \stackrel{\Delta}{=} F(x_1, x_2^*) = F(x_1, G_2(x_1)) = (x_1)^2 + x_1 \left(-\frac{1}{2} x_1 \right) + \left(-\frac{1}{2} x_1 \right)^2$$

$$F_1(x_1) = \frac{3}{4} x_1^2$$

Multistage optimization

Step 2. $$x_1^* = \longrightarrow F_1(x_1) = \min_{x_1}\left\{\frac{3}{4}x_1^2\right\}$$

$$\frac{d}{dx_1}F_1(x_1) = 2\frac{3}{4}x_1 = 0 \Rightarrow x_1^* = 0$$

Now we may return to initial condition:

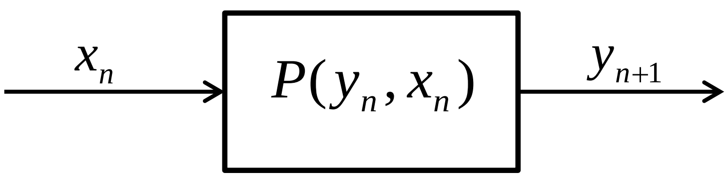
$$x_1^* = 0$$

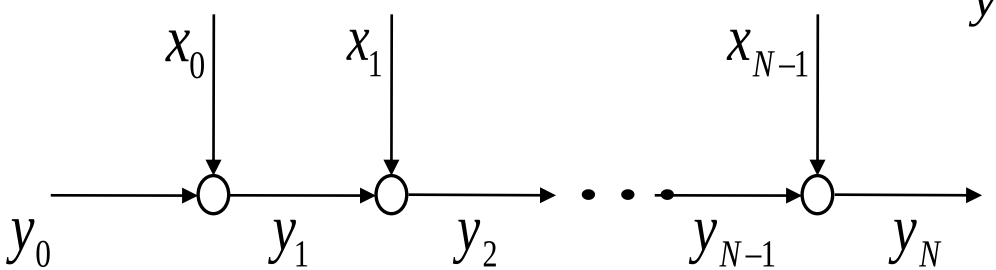
$$x_2^* = G_2(x_1^*) = -\frac{1}{2}x_1^* = 0$$

DYNAMIC PROGRAMMING

Dynamic programming

Dynamic process: $y_{n+1} = P(y_n, x_n),\ y_o$





n – the time step $n = 1, 2, \ldots, N$

x_n – decision made in the *n*–th step

y_n – state of the process in the *n*–th step

Decision making task: to determine a sequence of decisions: $x_0^*, x_1^*, \ldots, x_{N-1}^*,$

After N steps we want to achieve some goal, e.g.: $y_N = y^*$ – desired state,

The performance index $Q\left(x_0, \ldots, x_{N-1}, y_1, \ldots, y_N\right)$ must take minimal value

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Dynamic programming

Let us evaluate the quality of the sequence of decisions and their effects

$$Q(x_0, x_1, \ldots, x_{N-1}, y_1, y_2, \ldots, y_N)$$

Examples.:

$$Q(x_0, x_1, \ldots, x_{N-1}, y_1, y_2, \ldots, y_N) = \sum_{n=0}^{N-1} x_n^2, \quad y_N = y^*$$

$$Q(x_0, x_1, \ldots, x_{N-1}, y_1, y_2, \ldots, y_N) = \sum_{n=0}^{N-1} x_n^2, \quad \underline{y} \le y_N \le \overline{y}$$

$$Q(x_0, x_1, \ldots, x_{N-1}, y_1, y_2, \ldots, y_N) = \sum_{n=1}^{N-1} (y_n - y_n^*), \quad 0 \le x_n \le \overline{x}$$

$$Q(x_0, x_1, \ldots, x_{N-1}, y_1, y_2, \ldots, y_N) = \sum_{n=0}^{N-1} A_{n+1}(x_n, y_{n+1}), \quad \text{with constraints } x_n \text{ and } y_n$$

Dynamic programming

The following procedure may be applied:

$$y_0$$

$$y_1 = P(y_0, x_0)$$

$$y_2 = P(y_1, x_1) = P(P(y_0, x_0), x_1) \overset{\Delta}{=} P_1(y_0, x_0, x_1)$$

$$\vdots$$

$$y_N = P(y_{N-1}, x_{N-1}) = P(P(y_{N-2}, x_{N-2}), x_{N-1}) \overset{y_{n+1}=P(y_n, x_n)}{=} \cdots = P_{N-1}(y_0, x_0, x_1, \ldots, x_N)$$

After substitution to *Q(.)* we obtain:

$$Q(x_0, x_1, \ldots, x_{N-1}, y_1, y_2, \ldots, y_N) = \sum_{n=0}^{N-1} A_{n+1}(x_n, y_{n+1}) \overset{\Delta}{=} F(y_0, x_0, x_1, \ldots, x_{N-1})$$

Dynamic programming

Due to the fact, that:

$$Q(x_0, x_1, \ldots, x_{N-1}, y_1, y_2, \ldots, y_N) = \sum_{n=0}^{N-1} A_{n+1}(x_n, y_{n+1}) \overset{\Delta}{=} F(y_0, x_0, x_1, \ldots, x_{N-1})$$

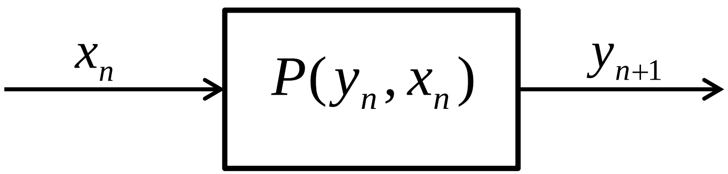
The optimization task:

$$x_0^*, x_1^*, \ldots, x_{N-1}^* \to Q(x_0^*, x_1^*, \ldots, x_{N-1}^*, y_1, y_2, \ldots, y_N) = \min_{x_0, x_1, \ldots, x_{N-1}} Q(x_0, x_1, \ldots, x_{N-1}, y_1, y_2, \ldots, y_N)$$

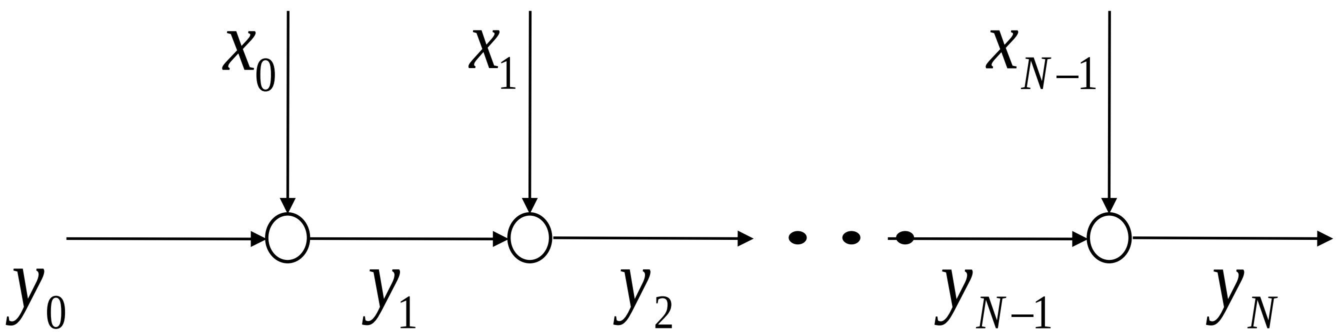
is equivalent to:

$$x_0^*, x_1^*, \ldots, x_{N-1}^* \to F(y_0, x_0^*, x_1^*, \ldots, x_{N-1}^*) = \min_{x_0, x_1, \ldots, x_{N-1}} F(y_0, x_0, x_1, \ldots, x_{N-1})$$

In order to solve the task above, the multi-stage approach may be applied.



$$y_{n+1} = P(y_n, x_n),\ y_o$$



$$Q(x_0, x_1, \ldots, x_{N-1}, y_1, y_2, \ldots, y_N) = \sum_{n=0}^{N-1} A_{n+1}(x_n, y_{n+1}) \stackrel{\Delta}{=} F(y_0, x_0, x_1, \ldots, x_{N-1})$$

Dynamic programming

$$Q(x_0, x_1, \ldots, x_{N-1}, y_1, y_2, \ldots, y_N) = \sum_{n=0}^{N-1} A_{n+1}(x_n, y_{n+1}) \stackrel{\Delta}{=} F(y_0, x_0, x_1, \ldots, x_{N-1})$$

Remark:

The final state y_N depends on decision x_{N-1} and the state y_{N-1}, which is dependent on previous decisions $x_0, x_1, \ldots, x_{N-2}$.

The previous state y_{N-1} depends on decision x_{N-2} and the state y_{N-2}, which is dependent on previous decisions $x_0, x_1, \ldots, x_{N-3}$.

⋮

The state y_2 depends on decision x_1 and the state y_1, which is dependent on previous decisions x_0, x_1.

The state y_1 depends on decision x_0 and the state y_0, which values are known.

In order to solve the task above, the multi-stage approach may be applied. Taking into account the form of performance index (sum of functions of decisions x_n resulting states y_{n+1}). Beginning from optimization of the last term we relate the solution from the previous state and previous decisions.

Dynamic programming

Step 1. $$x_{N-1}^{*} \rightarrow \min_{x_{N-1}} A_N\left(x_{N-1}, y_N\right)$$

We know, that: $$y_N = P\left(y_{N-1}, x_{N-1}\right)$$

$$x_{N-1}^{*} = G_{N-1}\left(y_{N-1}\right) \rightarrow \min_{x_{N-1}} A_N\left(x_{N-1}, P\left(y_{N-1}, x_{N-1}\right)\right)$$

$$V_{N-1}\left(y_{N-1}\right) \stackrel{\Delta}{=} \min_{x_{N-1}} A_N\left(x_{N-1}, P\left(y_{N-1}, x_{N-1}\right)\right) =$$

$$= A_N\left(x_{N-1}^{*}, P\left(y_{N-1}, x_{N-1}^{*}\right)\right) = A_N\left(G_{N-1}\left(y_{N-1}\right), P\left(y_{N-1}, G_{N-1}\left(y_{N-1}\right)\right)\right)$$

Dynamic programming

Step 2. $x^*_{N-2} \rightarrow \min_{x_{N-2}} \left\{ A_{N-1}\left(x_{N-2}, y_{N-1}\right) + V_{N-1}\left(y_{N-1}\right) \right\}$

We know, that $y_{N-1} = P\left(y_{N-2}, x_{N-2}\right)$

$$x^*_{N-2} = G_{N-2}\left(y_{N-2}\right) \rightarrow \min_{x_{N-2}} \left\{ A_{N-1}\left(x_{N-2}, P\left(y_{N-2}, x_{N-2}\right)\right) + V_{N-1}\left(P\left(y_{N-2}, x_{N-2}\right)\right) \right\}$$

$$V_{N-2}\left(y_{N-2}\right) \overset{\Delta}{=} \min_{x_{N-2}} \left\{ A_{N-1}\left(x_{N-2}, P\left(y_{N-2}, x_{N-2}\right)\right) + V_{N-1}\left(P\left(y_{N-2}, x_{N-2}\right)\right) \right\} =$$

$$= \left\{ A_{N-1}\left(x^*_{N-2}, P\left(y_{N-2}, x^*_{N-2}\right)\right) + V_{N-1}\left(P\left(y_{N-2}, x^*_{N-2}\right)\right) \right\} =$$

$$= A_{N-1}\left(G_{N-2}\left(y_{N-2}\right), P\left(y_{N-2}, G_{N-2}\left(y_{N-2}\right)\right)\right) + V_{N-1}\left(P\left(y_{N-2}, G_{N-2}\left(y_{N-2}\right)\right)\right)$$

⋮

Dynamic programming

Step N-1. $x_1^* \to \min_{x_1} \{ A_2(x_1, y_2) + V_2(y_2) \}$

We know, that $y_2 = P(y_1, x_1)$

$$x_1^* = G_1(y_1) \to \min_{x_1} \{ A_2(x_1, P(y_1, x_1)) + V_2(P(y_1, x_1)) \}$$

$$V_1(y_1) \overset{\Delta}{=} \min_{x_1} \{ A_2(x_1, P(y_1, x_1)) + V_2(P(y_1, x_1)) \} =$$
$$= A_2(x_1^*, P(y_1, x_1^*)) + V_2(P(y_1, x_1^*))$$
$$= A_2(G_1(y_1), P(y_1, G_1(y_1))) + V_2(P(y_1, G_1(y_1)))$$

Dynamic programming

Step N. $$x_0^* \to \min_{x_0} \{ A_1(x_0, y_1) + V_1(y_1) \}$$

We know, that $$y_1 = P(y_0, x_0)$$

$$x_0^* = G_0(y_0) \to \min_{x_0} \{ A_1(x_0, P(y_0, x_0)) + V_1(P(y_0, x_0)) \}$$

y_0 is known and from now on successive decisions may be determined

$$x_0^*, x_1^*, \ldots, x_{N-1}^*,$$

$$x_0^* = G_0(y_0) \to y_1 = P(y_0, x_0^*)$$
$$x_1^* = G_1(y_1) \to y_2 = P(y_2, x_2^*)$$
$$\vdots$$
$$x_{N-2}^* = G_{N-2}(y_{N-2}) \to y_{N-1} = P(y_{N-2}, x_{N-2}^*)$$
$$x_{N-1}^* = G_{N-1}(y_{N-1}) \to y_N = P(y_{N-1}, x_{N-1}^*)$$

Dynamic programming

Example: $y_{n+1} = P(y_n, x_n) = 2y_n + x_n, \quad y_o = 0$

$$Q(x_0, x_1, y_1, y_2) = \sum_{n=0}^{1} A_{n+1}(x_n, y_{n+1}) = \sum_{n=0}^{1} \left(x_n^2 + (y_{n+1} - 5)^2\right) =$$

$$= \left(x_0^2 + (y_1 - 5)^2\right) + \left(x_1^2 + (y_2 - 5)^2\right)$$

Step1.

$$x_1^* \to \min_{x_1}\left(x_1^2 + (y_2 - 5)^2\right)$$

$$y_2 = 2y_1 + x_1$$

$$x_1^* = G_1(y_1) \to \min_{x_1}\left(x_1^2 + (2y_1 + x_1 - 5)^2\right)$$

$$2x_1^* + 2(2y_1 + x_1^* - 5) = 0 \Rightarrow x_1^* = G_1(y_1) = \frac{5}{2} - y_1$$

$$V_1(y_1) = \left(\frac{5}{2} - y_1\right)^2 + \left(2y_1 + \frac{5}{2} - y_1 - 5\right)^2 = 2\left(y_1 - \frac{5}{2}\right)^2$$

Dynamic programming

Step 2.

$$x_0^* \to \min_{x_0}\left\{\left(x_0^2+(y_1-5)^2\right)+2\left(y_1-\frac{5}{2}\right)^2\right\}$$

$$y_1=2y_0+x_0$$

$$x_0^*=G_0(y_0)\to \min_{x_0}\left\{\left(x_0^2+(2y_0+x_0-5)^2\right)+2\left(2y_0+x_0-\frac{5}{2}\right)^2\right\}$$

$$2x_0^*+2(2y_0+x_0^*-5)+4\left(2y_0+x_0-\frac{5}{2}\right)=0 \Rightarrow x_0^*=G_0(y_0)=\frac{15}{8}-\frac{3}{2}y_0=\frac{15}{8}$$

Now, we return back with calculations:

$$x_0^*=\frac{15}{8}\to y_1=2\times 0+\frac{15}{8}=\frac{15}{8}$$

$$x_1^*=\frac{5}{2}-\frac{15}{8}=\frac{5}{8}\to y_1=2\times 0+\frac{5}{8}=\frac{5}{8}$$

Thank you for attention