Computer Science

Jerzy Świątek

Systems Modelling and Analysis

Choose yourself and new technologies

L.19a. Multicriteria decision making

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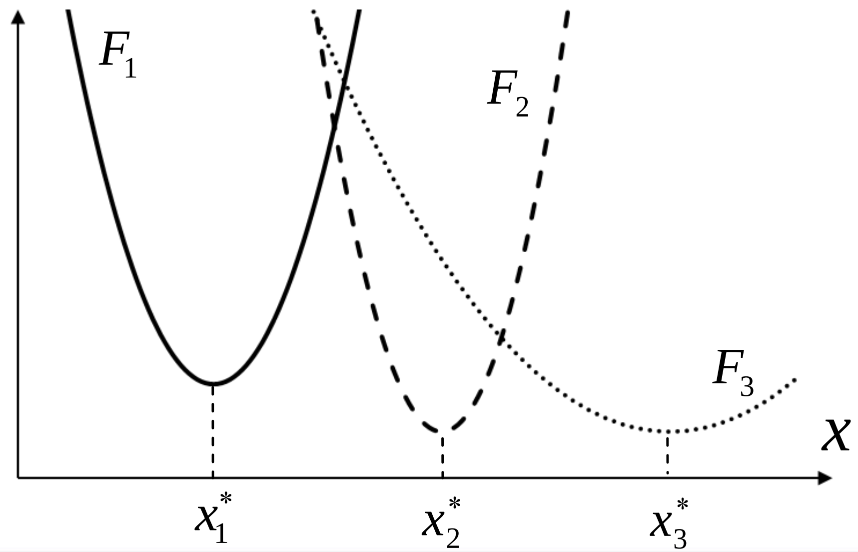
Multicriteria optimization

x – decision variables vector

$F_1(x), F_2(x), \dots, F_K(x)$ – performance indices



Multicriteria optimization

Synthetic performance index

H(.) – monotonic for all variables

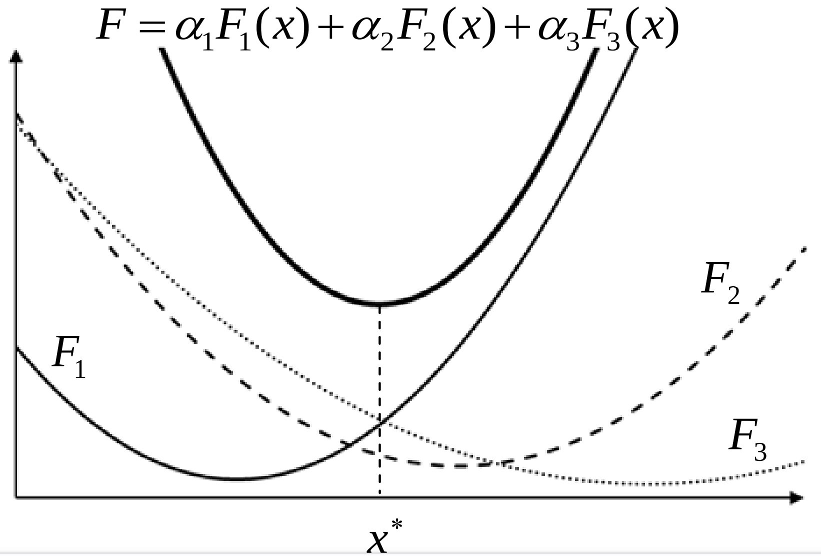
$$F(x) = H\big(F_1(x), F_2(x), \ldots, F_K(x)\big)$$

e. g.: $$F(x) = \sum_{k=1}^{K} \alpha_k F_k(x)$$

where: $$\sum_{k=1}^{K} \alpha_k = 1, \ \ \alpha_k > 0, \ \ k = 1,2,\ldots,K$$

$$F(x) = \prod_{k=1}^{K} F_k(x)$$

$$x^* \rightarrow F(x^*) = \min_{x \in D_x} F(x)$$



Multicriteria optimization

A selected performance index is optimized,

Upper limits for values of another performance indices are specified.

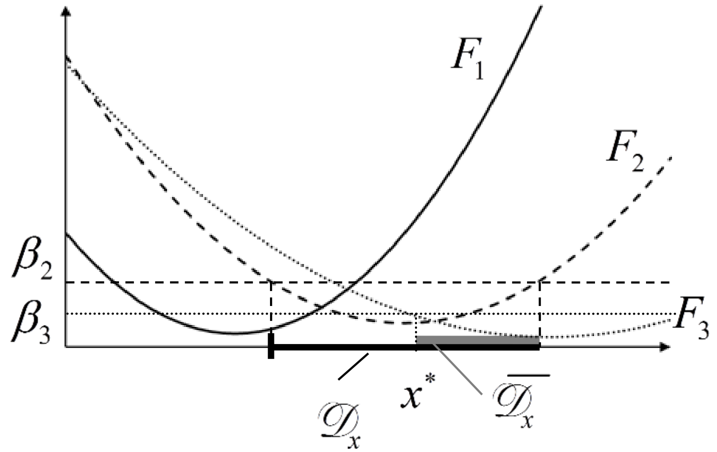
Let $F_1(x)$ be a selected performance index

$$F_k(x) \le \beta_k, \quad k = 2,3,\ldots,K$$

Requirements for performance indices are met

$$\overline{D_x} = D_x \cap \left\{x \in R^S : F_k(x) \le \beta_k,\; k = 2,\ldots,K\right\}$$

$$x^* \to F_1(x^*) = \min_{x \in \overline{D_x}} F_1(x)$$



Multicriteria optimization

Ranked/prioritized performance indices

$$F_1(x) \succ F_2(x) \succ \ldots \succ F_K(x) \quad x \in \mathrm{D}_x$$

Step 1. $\mathrm{D}_{x1} = \mathrm{D}_x$

$$x_1^* \to F_1(x_1^*) = \min_{x \in \mathrm{D}_{x1}} F_1(x)$$

Step 2. $\mathrm{D}_{x2} = \mathrm{D}_{x1} \cap \left\{ x \in \mathrm{R}^S : F_1(x) \le F_1\left(x_1^*\right) + \gamma_1 \right\}$

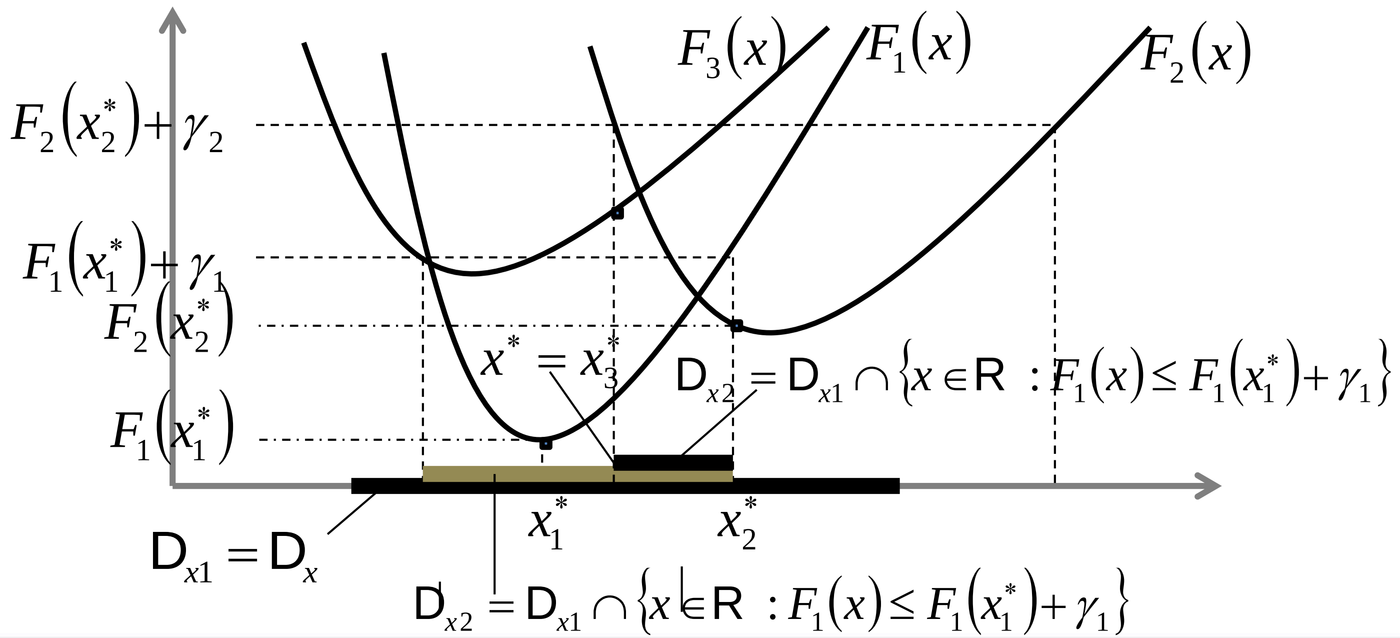
$$x_2^* \to F_2(x_2^*) = \min_{x \in \mathrm{D}_{x2}} F_2(x)$$

$\vdots$

Multicriteria optimization

Step K. $D_{xK} = D_{xK-1} \cap \left\{ x \in R^S : F_{K-1}(x) \le F_1\left(x^*_{K-1}\right) + \gamma_{K-1} \right\}$

$$x^*_K = x^*_K \to F_K(x^*_K) = \min_{x \in D_{xK}} F_K(x)$$



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Non-dominated solutions

Pareto-optimality

$$F_1(x), F_2(x), \ldots, F_K(x) \quad x \in D_x$$

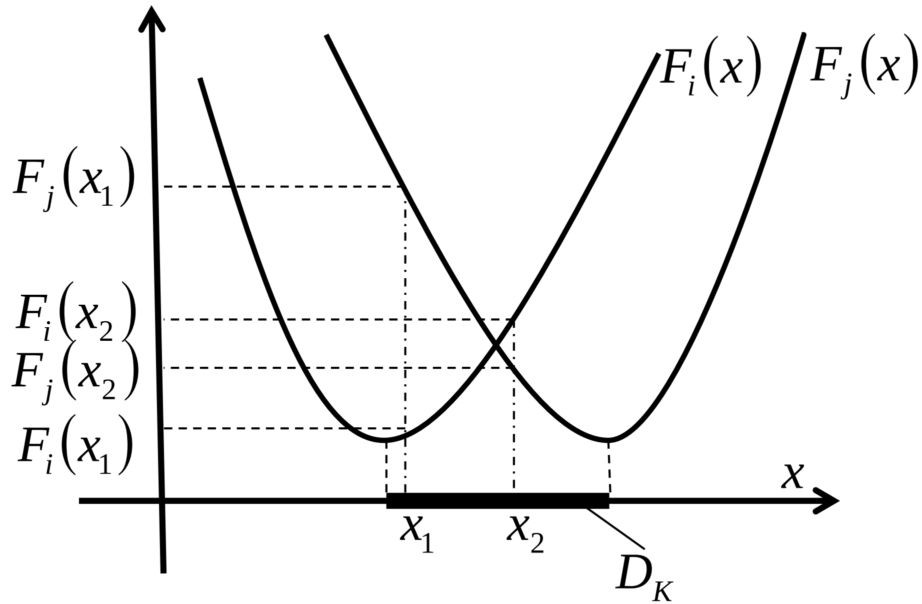
D_K – a set of non-dominated solutions

A set of non-dominated solutions $x \in D_x$ is a subset of such points $D_K \subseteq D_x$ for which performance indices has the following property: any of performance indices may decrease only if at least one of the remaining performance indices increases.

$$x_1, x_2 \in D_K \Leftrightarrow \forall i \in \{1, 2, \ldots, K\} \exists\, j \in \{1, 2, \ldots, K\}$$
$$F_i(x_1) < F_j(x_2) \Rightarrow F_j(x_1) < F_i(x_2)$$

Non-dominated solutions

$$x_1, x_2 \in D_K \Leftrightarrow \forall j \in \{1, 2, \ldots, K\} \; \exists\, i \in \{1, 2, \ldots, K\}$$
$$F_j(x_1) > F_j(x_2) \Rightarrow F_i(x_1) < F_i(x_2)$$



Non-dominated solutions

Analytical condition $$\sum_{k=1}^{K}\eta_k \nabla_x F_k(x) = 0_S \quad \eta_k \geq 0 \quad k = 1, 2, \ldots, K$$

Example:

$$F_1(x) = \left(x^{(1)} - 1\right)^2 + \left(x^{(2)} - 1\right)^2, \quad F_2(x) = \left(x^{(1)} + 1\right)^2 + \left(x^{(2)} + 1\right)^2$$

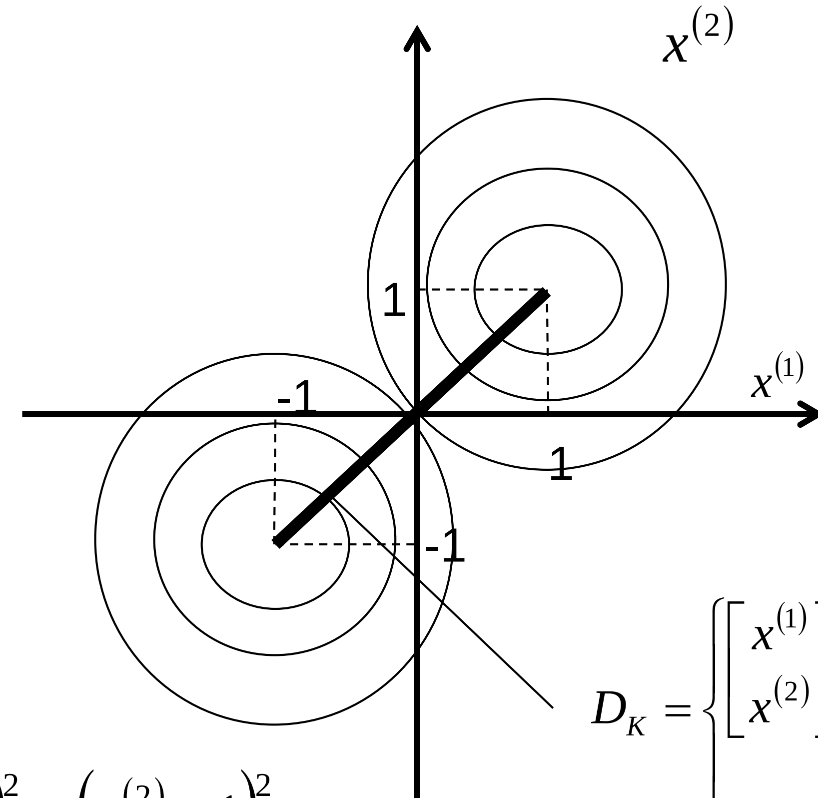
$$\eta_1 \begin{bmatrix} 2\left(x^{(1)} - 1\right) \\ 2\left(x^{(2)} - 1\right) \end{bmatrix} + \eta_2 \begin{bmatrix} 2\left(x^{(1)} + 1\right) \\ 2\left(x^{(2)} + 1\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad /2 \quad \eta_1, \eta_2 \geq 0$$

$$\eta_1 \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} + \eta_2 \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} = \eta_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \eta_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad /(\eta_1 + \eta_2)$$

$$\frac{\eta_1}{\eta_1 + \eta_2} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} + \frac{\eta_2}{\eta_1 + \eta_2} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} = \frac{\eta_1}{\eta_1 + \eta_2} \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \frac{\eta_2}{\eta_1 + \eta_2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} + = \lambda_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; gdzie \; \lambda_1 = \frac{\eta_1}{\eta_1 + \eta_2}, \lambda_2 = \frac{\eta_2}{\eta_1 + \eta_2}, \lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 \in [0, 1]$$

Non-dominated solutions

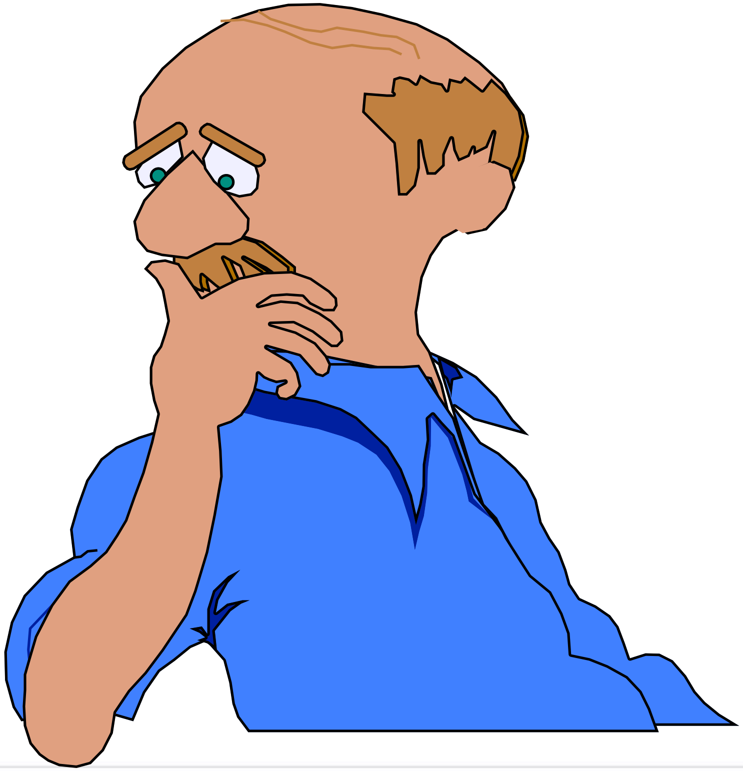


$$F_1(x) = \left(x^{(1)} - 1\right)^2 + \left(x^{(2)} - 1\right)^2$$

$$D_K = \left\{ \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \in R^2 : \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} + = \lambda_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; \lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 \in [0, 1] \right\}$$

$$F_2(x) = \left(x^{(1)} + 1\right)^2 + \left(x^{(2)} + 1\right)^2$$

Thank you for attention



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