

Computer Science

Jerzy Świątek

Systems Modelling and Analysis

Choose yourself and new technologies

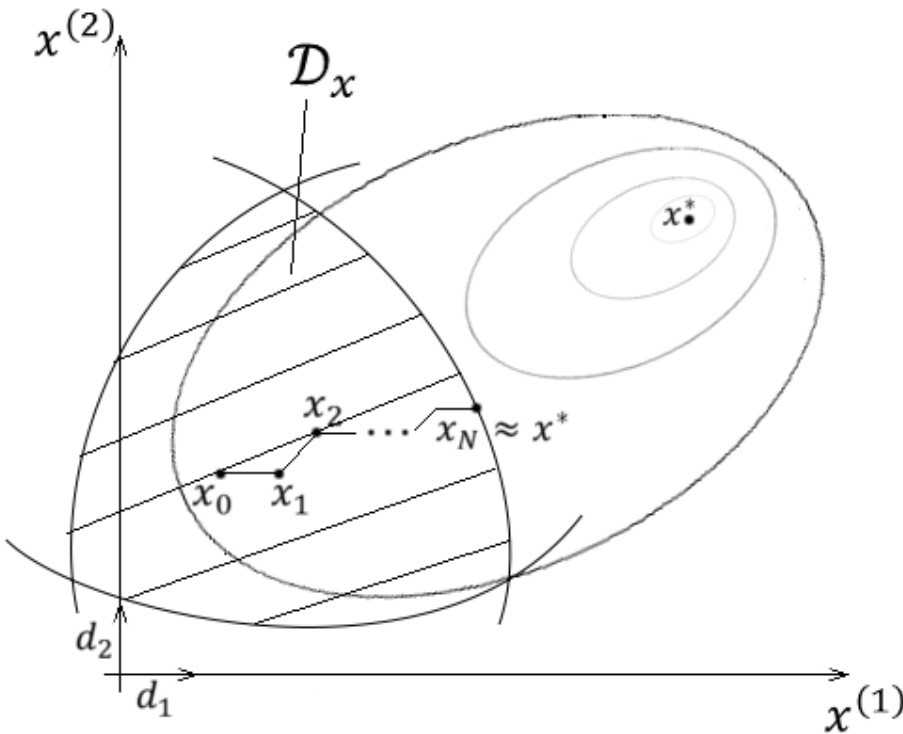
L.17.d Constrained optimization methods



Project co-financed from the EU European Social Fund



Numerical constrained optimization methods

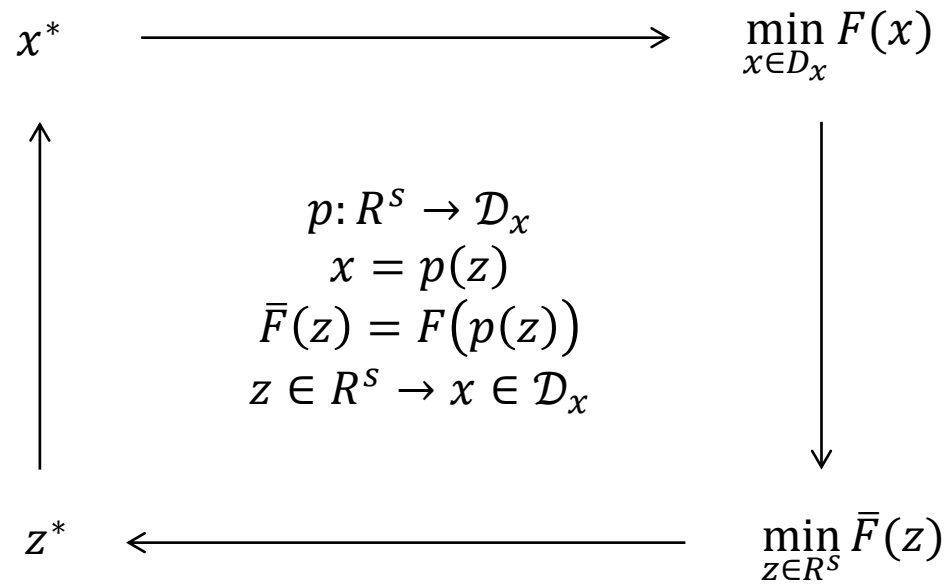


$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

1. Elimination of constraints
2. Penalty function method
 - exterior penalty
 - barrier function
3. Methods of feasible directions
4. Other approaches

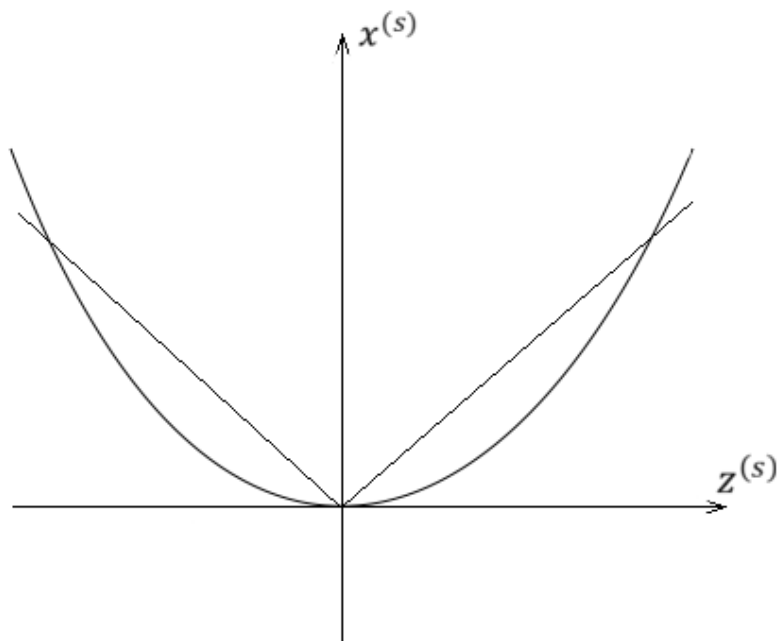


Elimination of constraints





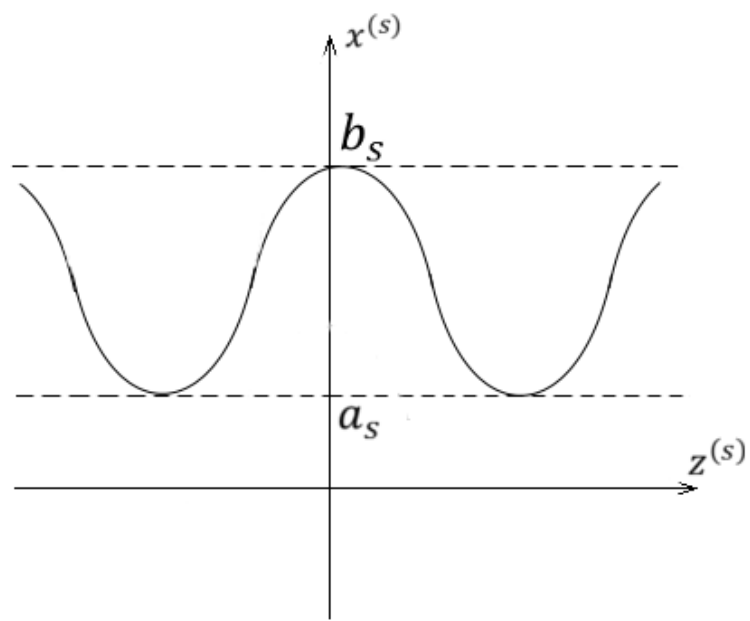
Example



$$D_x = \{x \in R^S, \quad x^{(s)} \geq 0, \quad s = 1, 2, \dots, S\}$$
$$x^{(s)} = (z^{(s)})^2 \text{ or } x^{(s)} = |z^{(s)}|$$
$$z^{(s)} \in R^S \rightarrow x^{(s)} \in [0, \infty)$$



Example



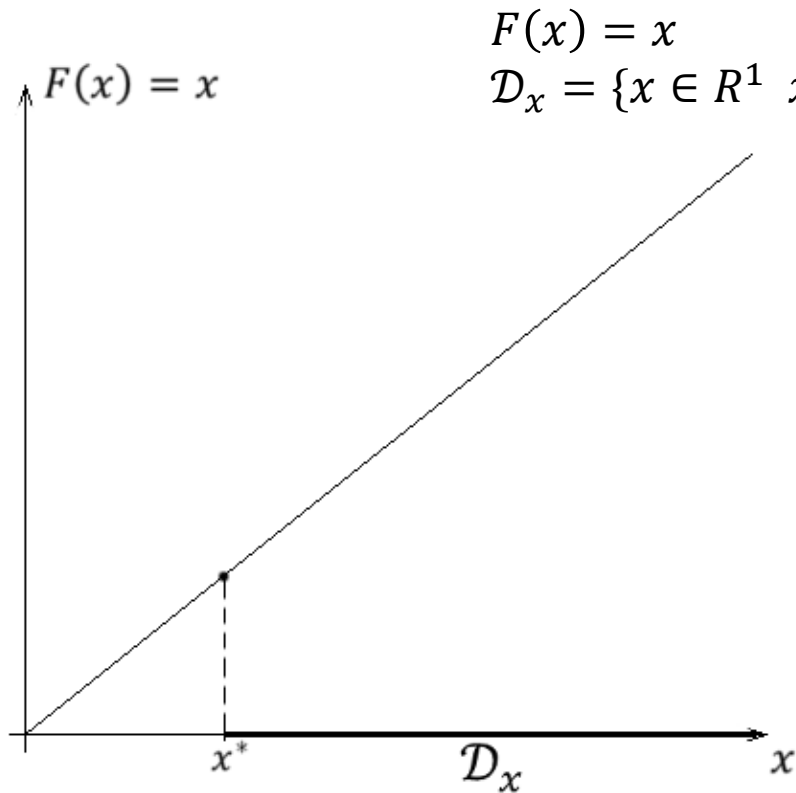
$$\mathcal{D}_x = \{x \in R^S, \quad a_s \leq x^{(s)} \leq b_s, \quad s = 1, 2, \dots, S\}$$

$$x^{(s)} = a_s + (b_s - a_s) \sin^2 z^{(s)}$$

$$z^{(s)} \in R^1 \rightarrow x^{(2)} \in [a_s, b_s]$$



Example



$$F(x) = x$$

$$D_x = \{x \in \mathbb{R}^1 \mid x \geq 1\}$$

$$x^* \rightarrow \min_{x \geq 1} x$$

$$x = z^2 + 1 = p(z)$$

$$z \in \mathbb{R} \rightarrow x \in [1, \infty)$$

$$\bar{F}(z) = z^2 + 1$$

$$z^* \rightarrow \min_{z \in \mathbb{R}} \bar{F}(z)$$

$$z^* \rightarrow \min_{z \in \mathbb{R}} (z^2 + 1)$$

$$(z^2 + 1)' = 2z = 0$$

$$z^* = 0$$

$$x^* = p(z^*) = (z^*)^2 + 1 = 1$$

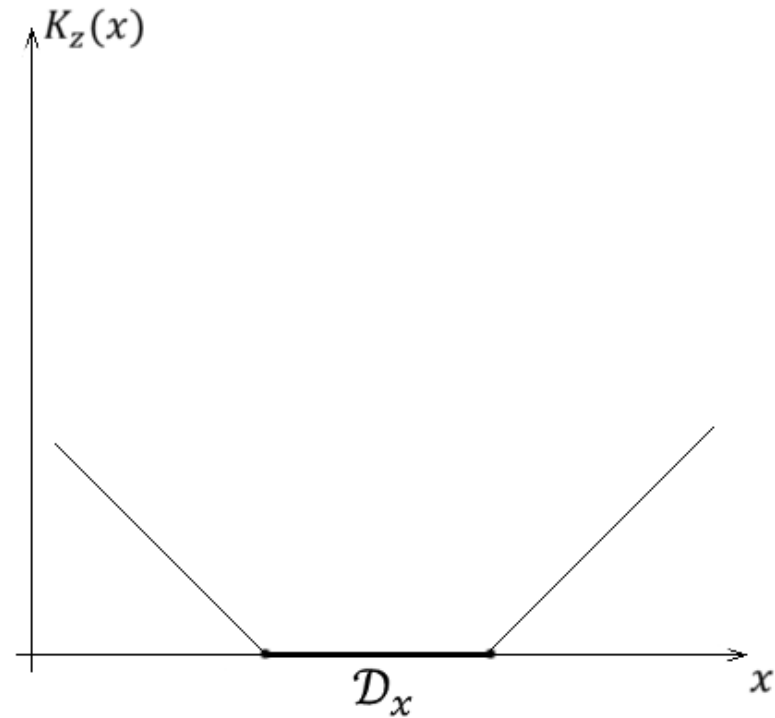


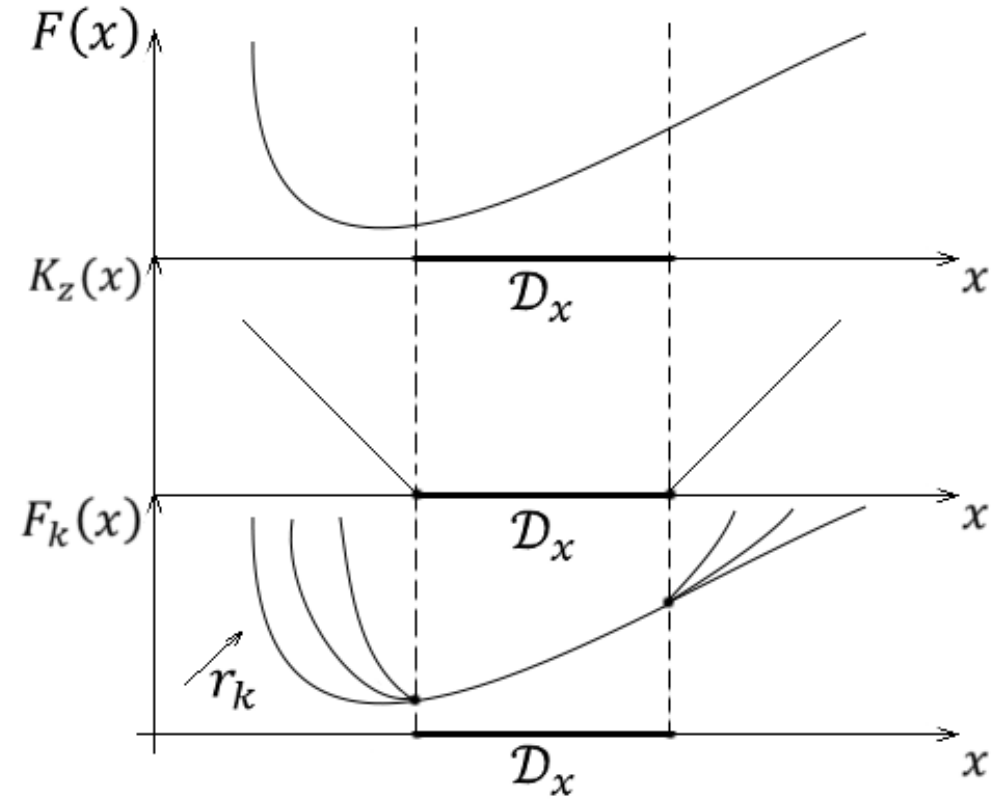
Penalty function method

Exterior penalty

$$F_k(x) = F(x) + r_k K_z(x)$$

$$K_z(x) \begin{cases} = 0 & x \in \mathcal{D}_x \\ > 0 & x \notin \mathcal{D}_x \end{cases}$$





$$r_k > 0$$

$$\lim_{k \rightarrow \infty} r_k = \infty$$



Example of exterior penalty function

$$\mathcal{D}_x = \{x \in R^s, \varphi_l(x) = 0, \quad l = 1, 2, \dots, L, \quad \psi_m(x) \leq 0, \quad m = 1, 2, \dots, M\}$$

$$\varphi_l(x) \rightarrow K_{lz}(x) = (\varphi_l(x))^2, \quad l = 1, 2, \dots, L$$

$$\psi_m(x) \rightarrow K_{mz}(x) = (\max\{0, \psi_m(x)\})^2, \quad m = 1, 2, \dots, M$$

$$K_w(x) = \sum_{l=1}^L r_l (\varphi_l(x))^2 + \sum_{m=1}^M \rho_m \max\{0, \varphi_m(x)\}$$



Example

$$F(x) = (x^{(1)})^2 + (x^{(2)})^2$$

$$\mathcal{D}_x = \{x \in \mathbb{R}^2, x^{(1)} + x^{(2)} - 2 = 0\}$$

$$F_k(x) = (x^{(1)})^2 + (x^{(2)})^2 + r_k(x^{(1)} + x^{(2)} - 2)^2$$

$$F'_k(x) = 0$$

$$2x^{(1)} + 2r_k(x^{(1)} + x^{(2)} - 2) = 0$$

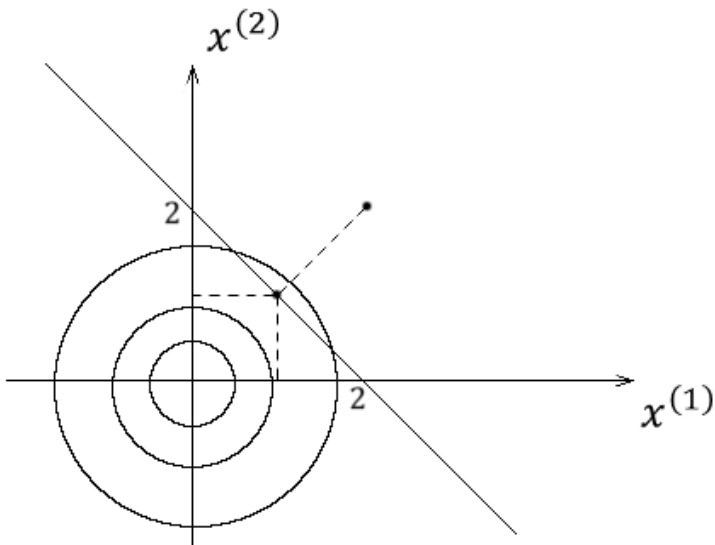
$$2x^{(2)} + 2r_k(x^{(1)} + x^{(2)} - 2) = 0$$

$$x^{(1)} = x^{(2)}$$

$$+2r_k(x^{(1)} + x^{(2)} - 2) = 0$$

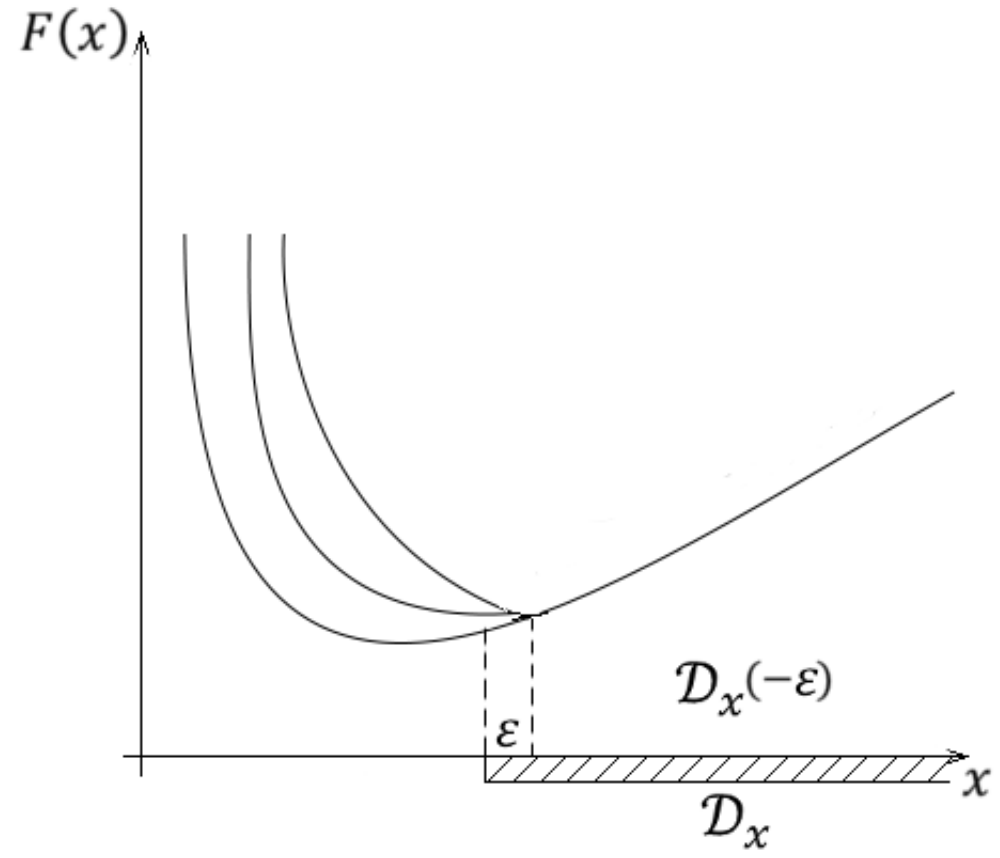
$$x^{(1)} = x^{(2)} = \frac{2r_k}{2r_k + 1}$$

$$x^{(1)} = x^{(2)} = \lim_{r_k \rightarrow \infty} \frac{2r_k}{2r_k + 1} = 1, \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$





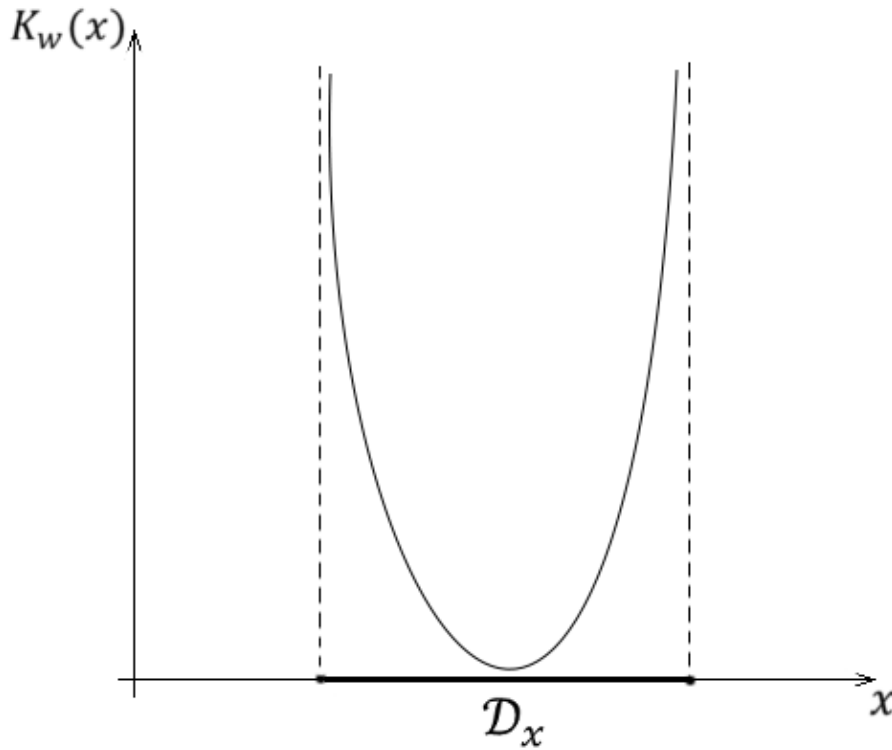
Modifications





Barrier function

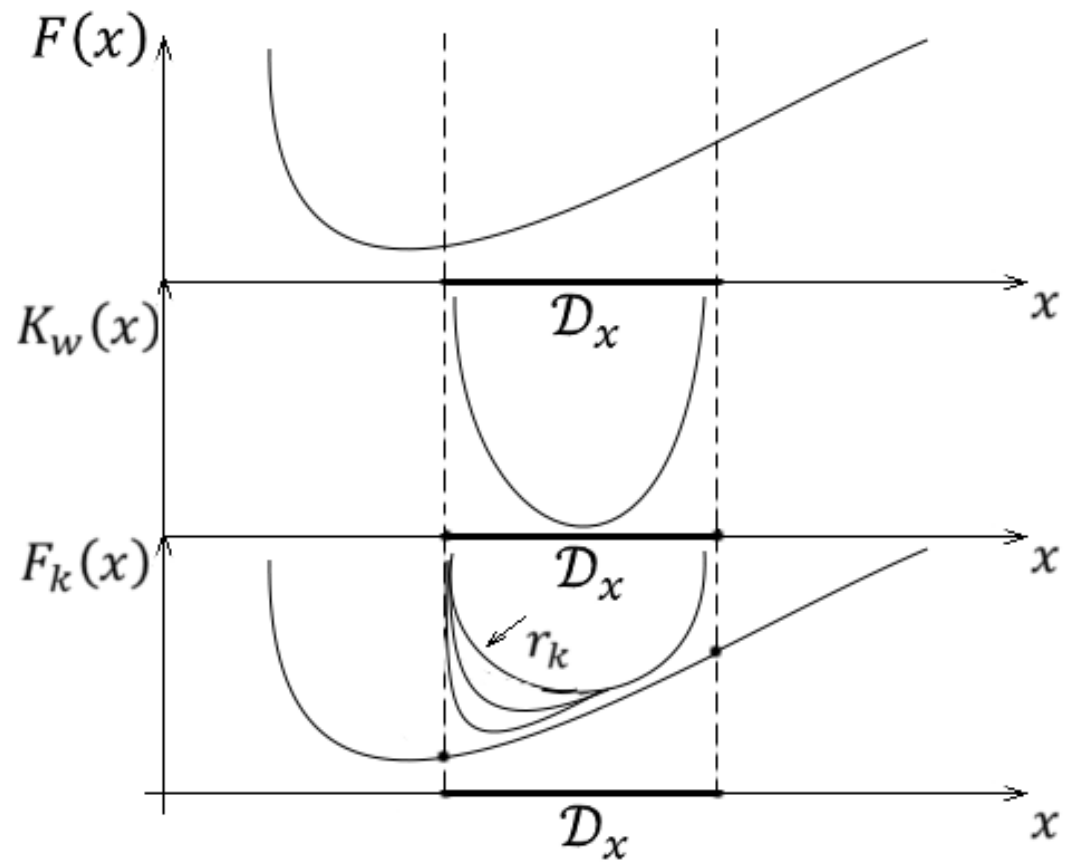
$$F_k(x) = F(x) + r_k K_w(x)$$



$K_w(x)$ – such a function, that

$$\exists x_1, x_2, \dots, x_n \in \mathcal{D}_x \quad \lim_{k \rightarrow \infty} x_k = x \in \mathcal{D}_x$$

$$\exists_k \quad K_w(x_k + 1) > K_w(x_k)$$



$$r_k > 0 \quad \lim_{k \rightarrow \infty} r_k = 0$$



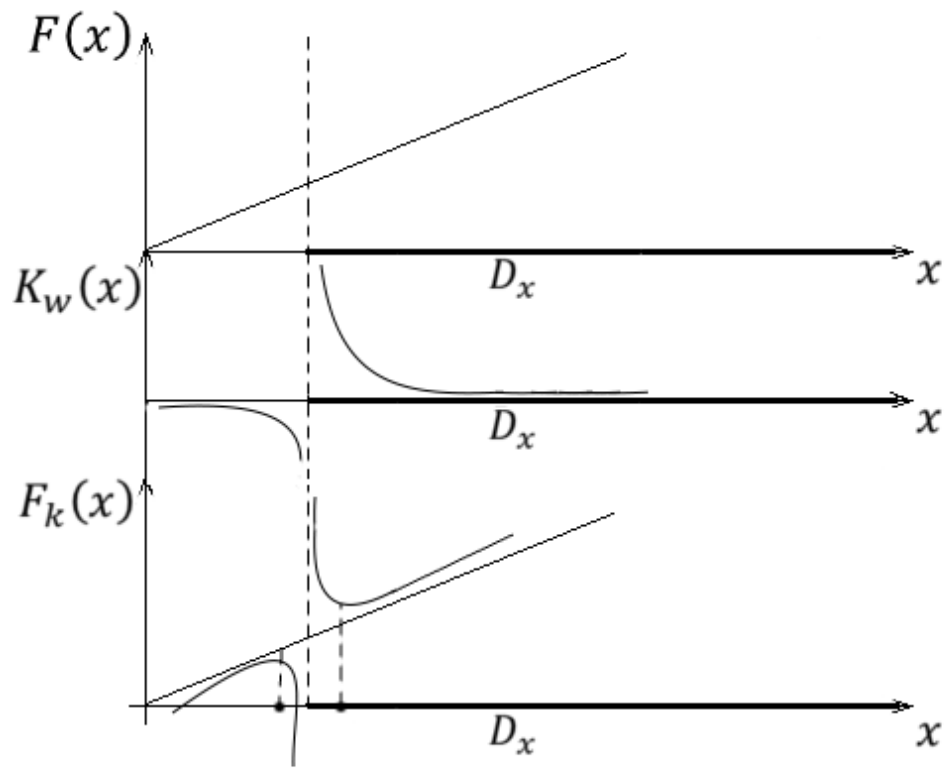
$$\mathcal{D}_x = \{x \in R^s, \quad \psi_m(x) \leq 0, \quad m = 1, 2, \dots, M\}$$

$$K_w(x) = \sum_{m=1}^M \frac{-r_m}{\psi_m(x)}, \quad m = 1, 2, \dots, M$$



Example

$$F(x) = x; \quad \mathcal{D}_x = \{x \in \mathbb{R}^1, x \geq 1\} \equiv \{x \in \mathbb{R}^1, 1 - x \leq 0\}$$



$$K_w(x) = \frac{-1}{1-x} = \frac{1}{x-1}$$

$$F_k(x) = x + r_k \frac{1}{x-1}$$

$$F'(x) = 1 + \frac{-r_k}{(x-1)^2} = 0$$

$$x_k = 1 \pm \sqrt{r_k}$$

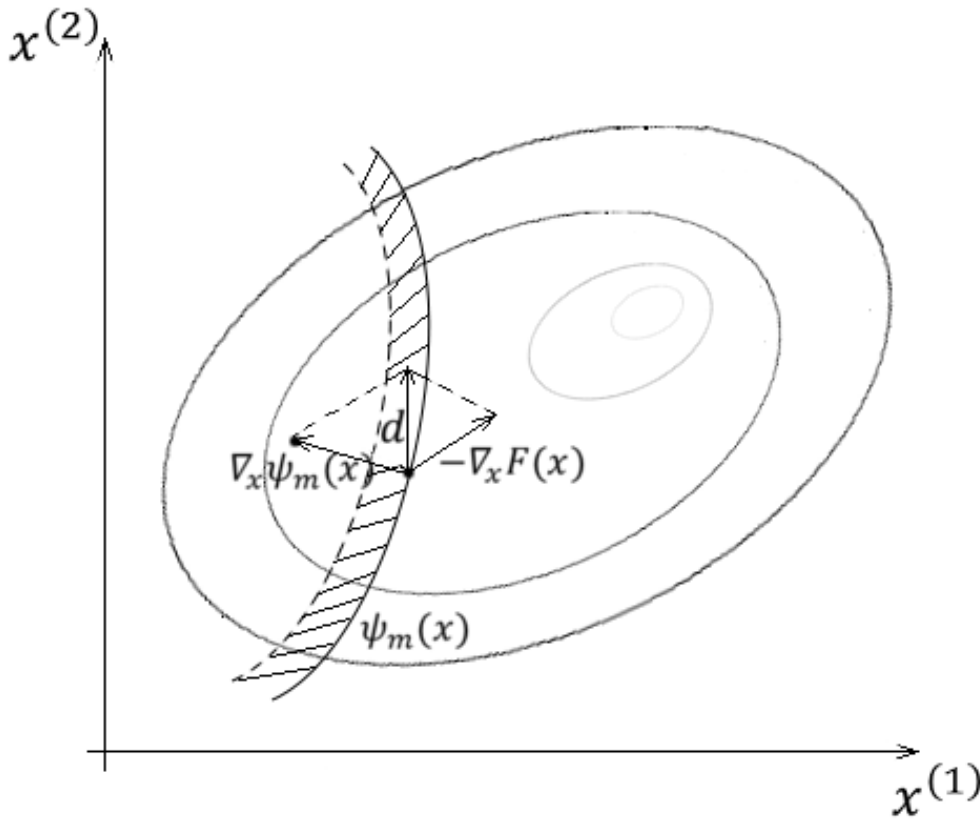
$$x_k = 1 - \sqrt{r_k} \notin \mathcal{D}_x$$

$$x_k = 1 + \sqrt{r_k} \in \mathcal{D}_x$$

$$\lim_{n \rightarrow \infty} x_n = 1$$



Feasible directions method

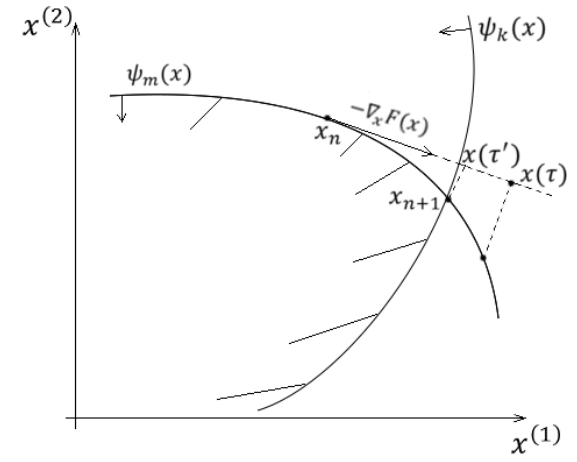
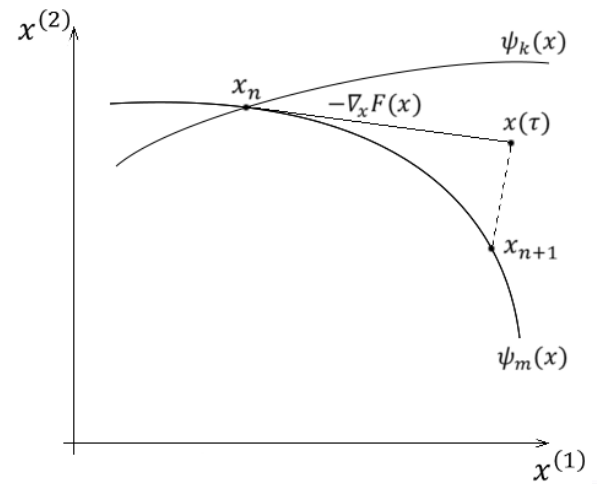
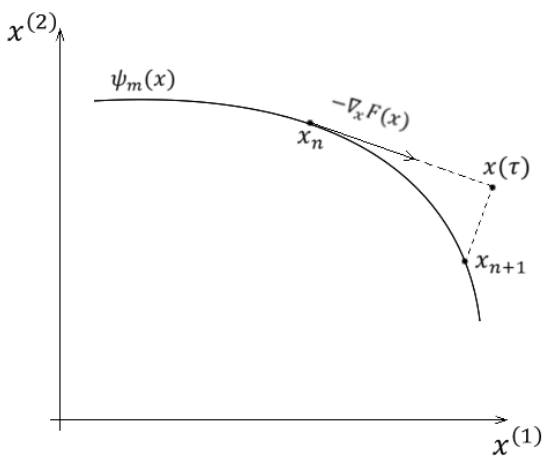


$$d = \frac{\nabla_x \psi_m(x)}{\|\nabla_x \psi_m(x)\|} - \frac{\nabla_x F(x)}{\|\nabla_x F(x)\|}$$

$$x: \psi(x) - \delta \leq 0$$

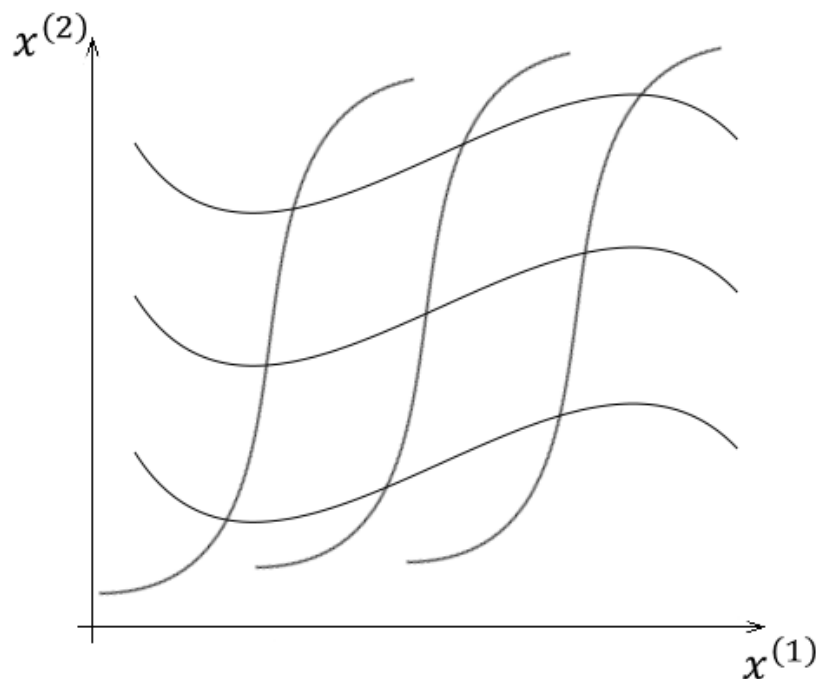


Gradient projection method of Rosen





The complex method





Random search

