

Computer Science

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Systems Modelling and Analysis

Choose yourself and new technologies

L.13. Identification of dynamic plants



HUMAN CAPITAL
HUMAN – BEST INVESTMENT!



Wrocław University of Technology

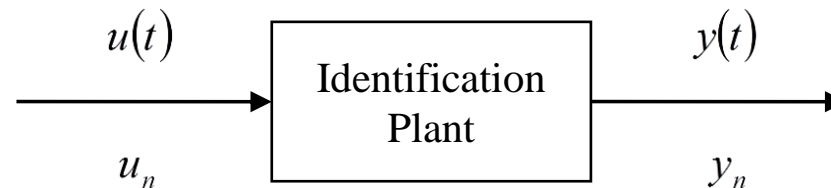
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Identification of Dynamic Plants



Model equation: $y_n + a_1 y_{n-1} + \dots + a_m y_{n-m} = b_0 u_n + b_1 u_{n-1} + \dots + b_k u_{n-k}$

or using transfer function: $A(z^{-1}) = a_1 z^{-1} + a_2 z^{-2} + \dots + a_m z^{-m}$

$$B(z^{-1}) = b_0 + b_1 z^{-1} + \dots + b_k z^{-k}$$

we can write equivalently: $y_n + A(z^{-1})y_n = B(z^{-1})u_n$

$$y_n(1 + A(z^{-1})) = B(z^{-1})u_n$$

$$K(z^{-1}) = \frac{B(z^{-1})}{(1 + K(z^{-1}))}$$



Identification of Dynamic Plants

Let us denote:

$$\theta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \\ b_0 \\ b_1 \\ \vdots \\ b_k \end{bmatrix} \quad X_n = \begin{bmatrix} -y_{n-1} \\ -y_{n-2} \\ \vdots \\ -y_{n-m} \\ u_n \\ u_{n-1} \\ \vdots \\ u_{n-k} \end{bmatrix}$$

and rewrite model equation as:

$$y_n = \theta^T X_n$$



Identification of Dynamic Plants

Least Squares method

Performance index:

$$Q_N(\theta) = \sum_{n=0}^N (y_n - \theta^T \mathbf{x}_n)^2$$

Optimization problem:

$$\theta_N \rightarrow Q_N(\theta_N) = \min_{\theta \in \Theta} Q_N(\theta) = \min_{\theta \in \Theta} \sum_{n=0}^N (y_n - \theta^T \mathbf{x}_n)^2$$

$$Q_N(\theta) = \sum_{n=0}^N (y_n - \theta^T \mathbf{x}_n)^2$$

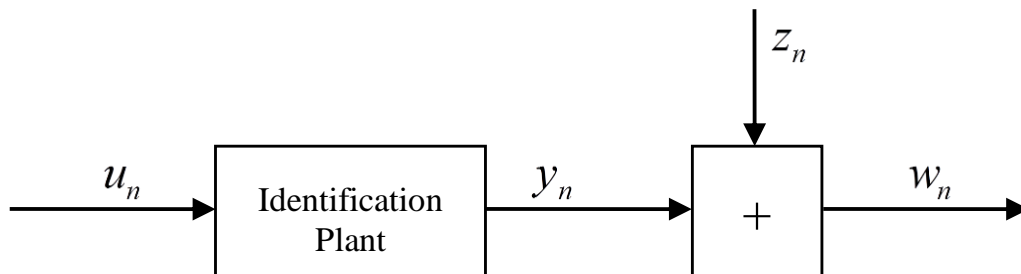
Solution:

$$\theta_N = \left(\sum_{n=0}^N \mathbf{x}_n \mathbf{x}_n^T \right)^{-1} \sum_{n=0}^N y_n \mathbf{x}_n$$



Identification of Dynamic Plants

Estimation of Linear Plant Parameters



$$y_n = a_1 y_{n-1} + \dots + a_m y_{n-m} = b_0 u_n + b_1 u_{n-1} + \dots + b_k u_{n-k}$$

Measurements with disturbances:

$$w_n = y_n + z_n$$

$$y_n = w_n - z_n$$

$$w_n + a_1 w_{n-1} + \dots + a_m w_{n-m} = b_0 u_n + b_1 u_{n-1} + \dots + b_k u_{n-k} + \underbrace{z_n + a_1 z_{n-1} + \dots + a_m z_{n-m}}_{v_n}$$

z is now correlated with previous states (we had $w = F(u, \theta) + z$ before)



Identification of Dynamic Plants

Estimation of Linear Plant Parameters

or by transfer function

$$A(z^{-1}) = a_1 z^{-1} + a_2 z^{-2} + \dots + a_m z^{-m}$$
$$B(z^{-1}) = b_0 + b_1 z^{-1} + \dots + b_k z^{-k}$$

we can rewrite equivalently:

$$w_n = A(z^{-1})w_n + B(z^{-1})u_n + v_n$$
$$w_n(1 + A(z^{-1})) = B(z^{-1})u_n$$

$$w_n = -a_1 w_{n-1} - a_2 w_{n-2} - \dots - a_m w_{n-m} + b_0 u_n + b_1 u_{n-1} + \dots + b_k u_{n-k} + v_n$$



Identification of Dynamic Plants

Estimation of Linear Plant Parameters

Let us denote:

$$\theta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \\ b_0 \\ b_1 \\ \vdots \\ b_k \end{bmatrix} \quad \mathbf{X}_n = \begin{bmatrix} -w_{n-1} \\ -w_{n-2} \\ \vdots \\ -w_{n-m} \\ u_n \\ u_{n-1} \\ \vdots \\ u_{n-k} \end{bmatrix}$$

and rewrite model equation as:

$$w_n = \theta^T \mathbf{X}_n + v_n$$

$$v_n = z_n + a_1 z_{n-1} + \dots + a_m z_{n-m}$$



Identification of Dynamic Plants

Estimation of Linear Plant Parameters

1. Direct approach

$$w_n = -A(z^{-1})w_n + B(z^{-1})u_n + v_n = -A(z^{-1})w_n + B(z^{-1})u_n + (1 + A(z^{-1}))z_n$$
$$w_n(1 - A(z^{-1})) = B(z^{-1})u_n + (1 + A(z^{-1}))z_n$$

$$\frac{w_n(1 + A(z^{-1}))}{1 + A(z^{-1})} = w_n = \frac{B(z^{-1})u_n}{1 + A(z^{-1})} + z_n$$

z is de-correlated, but parameters are nonlinearly involved in the equation.



Identification of Dynamic Plants

Estimation of Linear Plant Parameters

LSQ method, forgetting that are correlated:

Performance index:

$$Q_N(\theta) = \sum_{n=0}^N (w_n - \theta^T X_n)^2$$

$$\nabla_{\theta} Q_N(\theta) = -2 \sum_{n=0}^N (w_n - \theta^T X_n) X_n = 0_{m+k+1}$$

Solution:

$$\theta_N = \left(\sum_{n=0}^N X_n X_n^T \right)^{-1} \sum_{n=0}^N w_n X_n$$

But: v_n and z_n are correlated

$$v_n = z_n + a_1 z_{n-1} + \dots + a_m z_{n-m}$$



Identification of Dynamic Plants

Estimation of Linear Plant Parameters

2. De-correlation of disturbances

Let us assume, that $v_n = \rho v_{n-1} + z_n$

$$v_n (1 - \rho z^{-1}) = z_n$$

By multiplying both sides of equation:

$$w_n + a_1 w_{n-1} + \dots + a_m w_{n-m} = b_0 u_n + b_1 u_{n-1} + \dots + b_k u_{n-k} + \overbrace{z_n + a_1 z_{n-1} + \dots + a_m z_{n-m}}^{v_n}$$

by $(1 - \rho z^{-1})$, we obtain:

$$w_n (1 - \rho z^{-1}) = -a_1 w_{n-1} (1 - \rho z^{-1}) - \dots - a_m w_{n-m} (1 - \rho z^{-1}) + \\ + b_0 u_n (1 - \rho z^{-1}) + \dots + b_k u_{n-k} (1 - \rho z^{-1}) + v_n (1 - \rho z^{-1})$$



Identification of Dynamic Plants

Estimation of Linear Plant Parameters

Let $w_n^f = (1 - \rho z^{-1})w_n$, $u_n^f = (1 - \rho z^{-1})u_n$, where f stands for *filtered* . Then:

$$w_n^f = -a_1 w_{n-1}^f - \dots - a_m w_{n-m}^f + b_0 u_n^f + b_1 u_{n-1}^f + \dots + b_k u_{n-k}^f + z_n$$

and now we have grounds to apply LS method:

$$\theta_N = \left(\sum_{n=0}^N \mathbf{X}_n^f (\mathbf{X}_n^f)^T \right)^{-1} \sum_{n=0}^N w_n^f \mathbf{X}_n^f$$

where u_n^f and w_n^f are work out iteratively with use of following procedure:



Identification of Dynamic Plants

Estimation of Linear Plant Parameters

step 1.

$$\theta_N = \left(\sum_{n=0}^N \mathbf{X}_n \mathbf{X}_n^T \right)^{-1} \sum_{n=0}^N w_n \mathbf{X}_n$$

step 2.

$$w_n = \theta_N^T \mathbf{X}_n + v_n$$

$$v_n = w_n - \theta_N^T \mathbf{X}_n$$

step 3.

$$v_n = v_{n-1} \rho + z_n$$

$$\min_{\rho} \sum_{n=1}^N (v_n - \rho v_{n-1})^2 \Rightarrow \rho_N = \frac{\sum_{n=1}^N v_n v_{n-1}}{\sum_{n=1}^N v_{n-1}^2}$$



Identification of Dynamic Plants

Estimation of Linear Plant Parameters

step 4.

$$w_n^f = (1 - \rho_N z^{-1}) w_n = w_n - \rho_N w_{n-1}$$

$$u_n^f = (1 - \rho_N z^{-1}) u_n = u_n - \rho_N u_{n-1}$$

step 5.

$$\theta_N^f = \left(\sum_{n=0}^N \mathbf{X}_n^f (\mathbf{X}_n^f)^T \right)^{-1} \sum_{n=0}^N w_n^f \mathbf{X}_n^f$$

GOTO 2



Identification of Dynamic Plants

Estimation of Linear Plant Parameters

3. Generalized Least Square Method

$$(d_1 z^{-1} + d_2 z^{-2} + \dots + d_r z^{-r}) D(z^{-1}), \text{ where } D \text{ is matrix}$$

It means that we take into account correlation (back in time) between disturbances, not only ρz^{-1} , as before.

$$v_n = D(z^{-1})v_n + z_n$$

$$v_n = d_1 v_{n-1} + d_2 v_{n-2} + \dots + d_r v_{n-r} + z_n$$

$$v_n (I - D(z^{-1})) = z_n$$



Identification of Dynamic Plants

Estimation of Linear Plant Parameters

3. Generalized Least Square Method

By multiplying both sides of equation:

$$w_n + a_1 w_{n-1} + \dots + a_m w_{n-m} = b_0 u_n + b_1 u_{n-1} + \dots + b_k u_{n-k} + \overbrace{z_n + a_1 z_{n-1} + \dots + a_m z_{n-m}}^{v_n}$$

by $(1 - D(z^{-1}))$, we obtain:

$$w_n^f = w_n (1 - D(z^{-1})) \quad u_n^f = u_n (1 - D(z^{-1}))$$

$$w_n^f = -a_1 w_{n-1}^f - \dots - a_m w_{n-m}^f + b_0 u_n^f + \dots + b_m u_{n-m}^f + z_n$$

and iterative procedure is performed:



Identification of Dynamic Plants

Estimation of Linear Plant Parameters

3. Generalized Least Square Method

step 1.

$$\theta_N = \left(\sum_{n=0}^N \mathbf{X}_n \mathbf{X}_n^T \right)^{-1} \sum_{n=0}^N w_n \mathbf{X}_n$$

step 2.

$$v_n = w_n - \theta_N^T \mathbf{X}_n$$

step 3.

$$v_n = d_1 v_{n-1} + d_2 v_{n-2} + \dots + d_r v_{n-r} + z_n$$



Identification of Dynamic Plants

Estimation of Linear Plant Parameters

3. Generalized Least Square Method

By denoting $\Omega = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_r \end{bmatrix}$, $\bar{v}_n = \begin{bmatrix} v_{n-1} \\ v_{n-2} \\ \vdots \\ v_{n-r} \end{bmatrix}$ we can write equivalently:

$$v_n = \Omega^T \bar{v}_n + z_n$$

$$\Omega_N \rightarrow \min_{\Omega} \sum_{n=r}^N (v_n - \Omega^T \bar{v}_n)^2 \Rightarrow \Omega_N = \left(\sum_{n=r}^N \bar{v}_n \bar{v}_n^T \right)^{-1} \sum_{n=r}^N v_n \bar{v}_n$$

which gives us values of parameters of polynomial D .



Identification of Dynamic Plants

Estimation of Linear Plant Parameters

3. Generalized Least Square Method

step 4.

$$w_n^f = w_n (1 - D(z^{-1})) \quad \leftarrow \quad \Omega_N$$

$$u_n^f = u_n (1 - D(z^{-1})) \quad \leftarrow \quad \Omega_N$$

step 5.

$$\theta_N^f = \left(\sum_{n=0}^N \mathbf{X}_n^f (\mathbf{X}_n^f)^T \right)^{-1} \sum_{n=0}^N w_n^f X_n^f$$

GOTO 2



Identification of Dynamic Plants

Recursive Algorithms

$$\theta_N = \left(\sum_{n=1}^N \mathbf{x}_n \mathbf{x}_n^T \right)^{-1} \sum_{n=1}^N \mathbf{x}_n w_n$$

a new measurement \mathbf{x}_{N+1} , w_{N+1} comes in.

Update algorithm:

$$\theta_{N+1} = \Psi_R(\theta_N, \mathbf{x}_{N+1}, w_{N+1})$$

$$\theta_{N+1} = \left(\sum_{n=1}^{N+1} \mathbf{x}_n \mathbf{x}_n^T \right)^{-1} \sum_{n=1}^{N+1} \mathbf{x}_n w_n = \left(\sum_{n=1}^N \mathbf{x}_n \mathbf{x}_n^T + \mathbf{x}_{N+1} \mathbf{x}_{N+1}^T \right)^{-1} \left(\sum_{n=1}^N \mathbf{x}_n w_n + \mathbf{x}_{N+1} w_{N+1} \right)$$



Identification of Dynamic Plants

Useful Transformations:

$$(\mathbf{A} + \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} + \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1}$$

$\mathbf{B} = \mathbf{x}$ – column vector

$$\mathbf{D}^{-1} = 1$$

$$\mathbf{C} = \mathbf{x}^T$$

$$(\mathbf{A} + \mathbf{x}\mathbf{x}^T)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{x}(1 + \mathbf{x}^T\mathbf{A}^{-1}\mathbf{x})^{-1}\mathbf{x}^T\mathbf{A}^{-1}$$

$$\mathbf{D}^{-1} = -1$$

$$(\mathbf{A} - \mathbf{x}\mathbf{x}^T)^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{x}(1 - \mathbf{x}^T\mathbf{A}^{-1}\mathbf{x})^{-1}\mathbf{x}^T\mathbf{A}^{-1}$$

$$\mathbf{D}^{-1} = \alpha$$

$$(\mathbf{A} + \alpha\mathbf{x}\mathbf{x}^T)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{x}\left(\frac{1}{\alpha} + \mathbf{x}^T\mathbf{A}^{-1}\mathbf{x}\right)^{-1}\mathbf{x}^T\mathbf{A}^{-1}$$





Identification of Dynamic Plants

Recursive Algorithms

$$\begin{aligned} \left(\sum_{n=1}^{N+1} \mathbf{x}_n \mathbf{x}_n^T \right)^{-1} &= \left(\sum_{n=1}^N \mathbf{x}_n \mathbf{x}_n^T + \mathbf{x}_{N+1} \mathbf{x}_{N+1}^T \right)^{-1} = \\ &= \left(\sum_{n=1}^N \mathbf{x}_n \mathbf{x}_n^T \right)^{-1} - \left(\sum_{n=1}^N \mathbf{x}_n \mathbf{x}_n^T \right)^{-1} \frac{\mathbf{x}_{N+1} \mathbf{x}_{N+1}^T}{1 + \mathbf{x}_{N+1}^T \left(\sum_{n=1}^N \mathbf{x}_n \mathbf{x}_n^T \right)^{-1} \mathbf{x}_{N+1}} \left(\sum_{n=1}^N \mathbf{x}_n \mathbf{x}_n^T \right)^{-1} \end{aligned}$$

Let $P_N = \left(\sum_{n=1}^N \mathbf{x}_n \mathbf{x}_n^T \right)^{-1}$

$$P_{N+1} = P_N - P_N \frac{\mathbf{x}_{N+1} \mathbf{x}_{N+1}^T}{1 + \mathbf{x}_{N+1}^T P_N \mathbf{x}_{N+1}} P_N$$

We can rewrite previous equation as:





Identification of Dynamic Plants

Recursive Algorithms

$$\begin{aligned}
 \theta_{N+1} &= \left(P_N - P_N \frac{\mathbf{x}_{N+1} \mathbf{x}_{N+1}^T}{1 + \mathbf{x}_{N+1}^T P_N \mathbf{x}_{N+1}} P_N \right) \left(\sum_{n=1}^N \mathbf{x}_n w_n + \mathbf{x}_{N+1} w_{N+1} \right) = \\
 &= P_N \sum_{n=1}^N \mathbf{x}_n w_n + P_N \mathbf{x}_{N+1} w_{N+1} - \frac{P_N \mathbf{x}_{N+1} \mathbf{x}_{N+1}^T P_N}{1 + \mathbf{x}_{N+1}^T P_N \mathbf{x}_{N+1}} \sum_{n=1}^N \mathbf{x}_n w_n - \frac{P_N \mathbf{x}_{N+1} \mathbf{x}_{N+1}^T P_N \mathbf{x}_{N+1} w_{N+1}}{1 + \mathbf{x}_{N+1}^T P_N \mathbf{x}_{N+1}} = \\
 &= P_N \sum_{n=1}^N \mathbf{x}_n w_n + \frac{P_N \mathbf{x}_{N+1} w_{N+1} + P_N \mathbf{x}_{N+1} w_{N+1} \mathbf{x}_{N+1}^T P_N \mathbf{x}_{N+1} - P_N \mathbf{x}_{N+1} w_{N+1} \mathbf{x}_{N+1}^T P_N \mathbf{x}_{N+1} - P_N \mathbf{x}_{N+1} \mathbf{x}_{N+1}^T P_N \sum_{n=1}^N \mathbf{x}_n w_n}{1 + \mathbf{x}_{N+1}^T P_N \mathbf{x}_{N+1}} = \\
 &= P_N \sum_{n=1}^N \mathbf{x}_n w_n + \frac{P_N \mathbf{x}_{N+1} w_{N+1} - P_N \mathbf{x}_{N+1} \mathbf{x}_{N+1}^T P_N \sum_{n=1}^N \mathbf{x}_n w_n}{1 + \mathbf{x}_{N+1}^T P_N \mathbf{x}_{N+1}} = \\
 &= \theta_N + K_{N+1} (w_{N+1} - \mathbf{x}_{N+1}^T \theta_N)
 \end{aligned}$$

where: $K_{N+1} = \frac{P_N \mathbf{x}_{N+1}}{1 + \mathbf{x}_{N+1}^T P_N \mathbf{x}_{N+1}}$



Identification of Dynamic Plants

Recursive Algorithms

Taking everything together, we obtain the *on-line identification* algorithm

$$\theta_{N+1} = \theta_N + K_{N+1} (w_{N+1} - \mathbf{x}_{N+1}^T \theta_N)$$

$$K_{N+1} = \frac{P_N \mathbf{x}_{N+1}}{1 + \mathbf{x}_{N+1}^T P_N \mathbf{x}_{N+1}}$$

$$P_{N+1} = P_N - \frac{P_N \mathbf{x}_{N+1} \mathbf{x}_{N+1}^T P_N}{1 + \mathbf{x}_{N+1}^T P_N \mathbf{x}_{N+1}}$$



Identification of Dynamic Plants

Time-Variant Plants – Parameters Change With Time

1. Including previous measurements with weights

$$\theta_N \rightarrow \min \sum_{n=1}^N \beta_n (w_n - \theta^T x_n)^2$$

How to choose values of β_n ?

- When n is small β should be also small (close to zero).
- When n is large β should be also large (close to N).



Identification of Dynamic Plants

Time-Variant Plants – Parameters Change With Time

$$\beta_n = \alpha^{N-n}, \quad \alpha \in (1/2, 1)$$

Optimization problem:

$$\theta_N \rightarrow \min \sum_{n=1}^N \alpha^{N-n} (w_n - \theta^T x_n)^2$$

Solution:

$$\theta_N = \left(\sum_{n=1}^N \alpha^{N-n} x_n x_n^T \right)^{-1} \sum_{n=1}^N \alpha^{N-n} x_n w_n$$

$$\theta_{N+1} = \left(\sum_{n=1}^{N+1} \alpha^{N+1-n} x_n x_n^T \right)^{-1} \sum_{n=1}^{N+1} \alpha^{N+1-n} x_n w_n =$$

$$= \left(\alpha \sum_{n=1}^N \alpha^{N-n} x_n x_n^T + x_{N+1} x_{N+1}^T \right)^{-1} \left(\alpha \sum_{n=1}^N \alpha^{N-n} x_n w_n + x_{N+1} w_{N+1} \right)$$

⋮



Identification of Dynamic Plants

Time-Variant Plants – Parameters Change With Time

Taking everything together, we obtain the *identification* algorithm

$$\theta_{N+1} = \theta_N + K_{N+1} (w_{N+1} - \mathbf{x}_{N+1}^T \theta_N)$$

$$K_{N+1} = \frac{P_N \mathbf{x}_{N+1}}{\alpha + \mathbf{x}_{N+1}^T P_N \mathbf{x}_{N+1}}$$

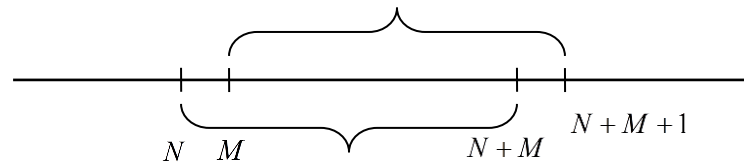
$$P_{N+1} = \frac{1}{\alpha} \left(P_N - \frac{P_N \mathbf{x}_{N+1} \mathbf{x}_{N+1}^T P_N}{1 + \mathbf{x}_{N+1}^T P_N \mathbf{x}_{N+1}} \right)$$



Identification of Dynamic Plants

Shifting window

We use only M last measurements for identification. After getting new measurement, the last one is thrown out.



$$\theta_{N+M,M}$$

$\theta_{N+M+1,M+1}$ add measurement $\mathbf{x}_{N+M+1}, w_{N+M+1}$

$\theta_{N+M+1,M}$ reject measurement \mathbf{x}_N, w_N



Identification of Dynamic Plants

Shifting window

Optimization problem:

$$\theta_{N+M,M} \rightarrow \min_{\theta} \sum_{n=N}^{N+M} (w_n - \theta^T x_n)^2$$

Identification algorithm:

$$\theta_{N+M,M} = \left(\sum_{n=N}^{N+M} x_n x_n^T \right)^{-1} \sum_{n=N}^{N+M} x_n w_n$$



Identification of Dynamic Plants

Shifting window

step 1.

$$\theta_{N+M+1, M+1} = \left(\sum_{n=N}^{N+M+1} \mathbf{x}_n \mathbf{x}_n^T \right)^{-1} \sum_{n=N}^{N+M+1} \mathbf{x}_n w_n \quad (\text{add } \mathbf{x}_{N+M+1}, w_{N+M+1})$$

step 2.

$$\theta_{N+M+1, M} = \left(\sum_{n=N+1}^{N+M+1} \mathbf{x}_n \mathbf{x}_n^T \right)^{-1} \sum_{n=N+1}^{N+M+1} \mathbf{x}_n w_n \quad (\text{reject } \mathbf{x}_N, w_N)$$



Identification of Dynamic Plants

Shifting window

Let $P_{N+M,M} = \left(\sum_{n=N}^{N+M} \mathbf{x}_n \mathbf{x}_n^T \right)^{-1}$

step 1.

$$P_{N+M+1,M+1} = \left(\sum_{n=N}^{N+M+1} \mathbf{x}_n \mathbf{x}_n^T \right)^{-1}$$

$$\mathbf{v}_{N+M+1,M+1} = \boldsymbol{\theta}_{N+M,M} + K_{N+M+1,M+1} \left(w_{N+M+1} - \mathbf{x}_{N+M+1}^T \boldsymbol{\theta}_{N+M,M} \right)$$

$$K_{N+M+1,M+1} = \frac{P_{N+M,M} \mathbf{x}_{N+M+1}}{I + \mathbf{x}_{N+M+1}^T P_{N+M,M} \mathbf{x}_{N+M+1}}$$

$$P_{N+M+1,M+1} = P_{N+M,M} - \frac{P_{N+M,M} \mathbf{x}_{N+M+1} \mathbf{x}_{N+M+1}^T P_{N+M,M}}{I + \mathbf{x}_{N+M+1}^T P_{N+M,M} \mathbf{x}_{N+M+1}}$$



Identification of Dynamic Plants

Shifting window

step 2.

$$\theta_{N+M+1,M} = \theta_{N+M,M+1} - K_{N+M+1,M} \left(w_N - \mathbf{x}_N^T \theta_{N+M+1,M+1} \right)$$

$$K_{N+M+1,M} = \frac{P_{N+M+1,M+1} \mathbf{x}_N}{1 - \mathbf{x}_N^T P_{N+M+1,M+1} \mathbf{x}_N}$$

$$P_{N+M+1,M} = P_{N+M+1,M+1} + P_{N+M+1,M+1} \frac{\mathbf{x}_N \mathbf{x}_N^T}{1 - \mathbf{x}_N^T P_{N+M+1,M+1} \mathbf{x}_N} P_{N+M+1,M+1}$$



Thank you for attention

