Computer Science

Jerzy Świątek

Systems Modelling and Analysis

Choose yourself and new technologies

L.12. Identification of dynamic plants

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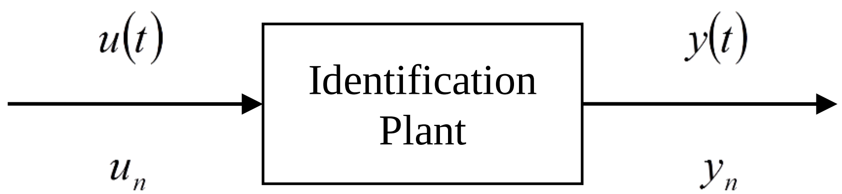
Identification of Dynamic Plants



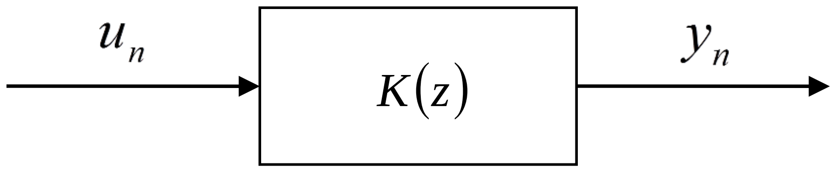
Descriptions:

- state variable;
- differential/difference equation;
- transfer functions: $K(s)$, $K(z)$;
- impulse response: $k_i(t)$, k_{in} ;
- step response: $h(t)$, $h_n(t)$.

Identification of Dynamic Plants

Identification of Impulse Responses

Discrete case



$$k_{in} = \mathrm{Z}^{-1}\left[K(z)\right]$$

$$y_n = \sum_{k=0}^{n} k_{ik} u_{n-k} = \sum_{k=0}^{n} k_{in-k} u_k$$ – discrete case

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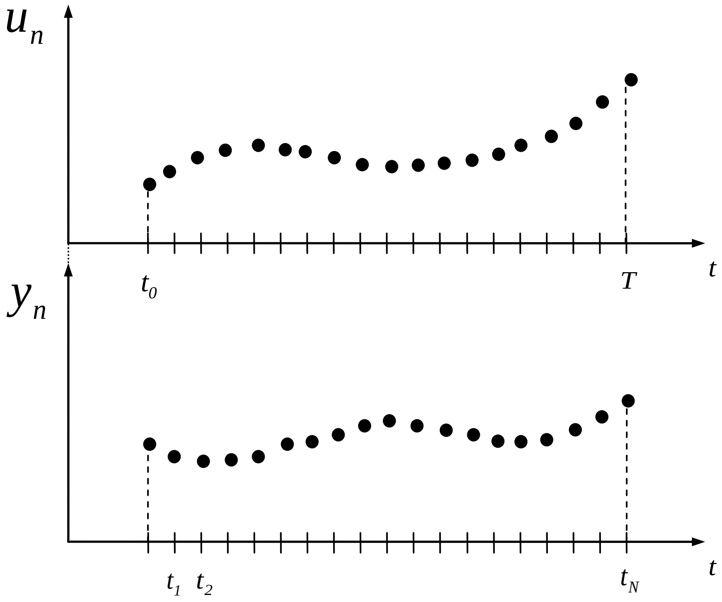
Identification of Dynamic Plants

For input signal u_n we measure respective output signal y_n :



$$t_0 < t_1 < \ldots < t_N \leq T$$

$$\{u_n\}_{n=0}^{N} \qquad \{y_n\}_{n=0}^{N}$$

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Identification of Dynamic Plants
Identification of Impulse Responses

Discrete case

We want to determine discrete input response (for $u_n = \delta_n$): k_{in}

We have sequence of measurements: $u_0, u_1, \ldots, u_N$
$y_0, y_1, \ldots, y_N$
and following tasks:

1) plant is linear: $k_{in}(\theta)$, but we don't know the values of parameters θ
2) we want to determine sequence of values of $k_{i0}, k_{i1}, \ldots, k_{iN}$

where:

k_{in} - impulse function

Identification of Dynamic Plants
Identification of Impulse Responses

Discrete case

3) plant is nonlinear, we approximate: $\bar{k}_{in}(\theta), \ \theta^*$

4) plant is nonlinear, we approximate by the sequence of discrete impulse response $k^*_{i0}, k^*_{i1}, \ldots, k^*_{iN}$

where:

$\bar{k}_{in}$ - given function

θ - unknown vector of parameter

Identification of Dynamic Plants
Identification of Impulse Responses

ad. 1. Linear plant

$$y_n = \sum_{k=0}^{N} k_{ik}(\theta) u_{n-k}$$

where:

k_{ik} - given function, θ - unknown

$$y_0 = k_{i0}(\theta) u_0$$
$$y_1 = k_{i0}(\theta) u_1 + k_{i1}(\theta) u_0$$
$$\vdots$$
$$y_N = k_{i0}(\theta) u_N + k_{i1}(\theta) u_{N-1} + \ldots + k_{iN}(\theta) u_0$$

$$N \cdot L \geq R$$
$$\dim y = L$$
$$\dim \theta = R$$

Solution of the above system of equations is calculated values of θ

Identification of Dynamic Plants
Identification of Impulse Responses

ad. 2. Determination of discrete impulse responses values $k_{i0}, k_{i1}, \ldots, k_{iN}$

$$y_n = \sum_{k=0}^{n} k_{ik} u_{n-k}$$

$$\begin{cases} y_0 = k_{i0} u_0 \\ y_1 = k_{i0} u_1 + k_{i1} u_0 \\ \vdots \\ y_N = k_{i0} u_N + k_{i1} u_{N-1} + \ldots + k_{iN} u_0 \end{cases}$$

Identification of Dynamic Plants
Identification of Impulse Responses

ad. 2. Determination of discrete impulse responses values $k_{i0}, k_{i1}, \dots, k_{iN}$

Denote:

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix} = Y_N \qquad \begin{bmatrix} u_0 & 0 & \cdots & 0 \\ u_1 & u_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ u_N & u_{N-1} & \cdots & u_0 \end{bmatrix} = U_N \qquad \begin{bmatrix} k_{i0} \\ k_{i1} \\ \vdots \\ k_{iN} \end{bmatrix} = \mathbf{k}_{iN}$$

Solution:

$$Y_N = U_N \mathbf{k}_{in} \Rightarrow \underline{\underline{\mathbf{k}_{in} = U_N^{-1} Y_N}}$$

Identification of Dynamic Plants
Identification of Impulse Responses

Ad. 3. Plant is nonlinear; approximation by $\bar{k}_{in}(\theta)$.

$$\bar{y}_n = \sum_{k=0}^{N} \bar{k}_{ik}(\theta) u_{n-k} \quad \text{– approximation}$$

where: $\bar{k}_{ik}$ - given function, θ - unknown vector of parameters

Performance index:

$$Q_N(\theta) = \sum_{n=0}^{N} (y_n - \bar{y}_n)^2 = \sum_{n=0}^{N} \left(y_n - \sum_{k=0}^{n} \bar{k}_{ik}(\theta) u_{n-k} \right)^2$$

Optimization problem:

$$\theta_N^* \to Q_N(\theta_N^*) = \min_{\theta \in \Theta} Q_N(\theta)$$

Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate by the sequence of discrete impulse responses $k^*_{i0}, k^*_{i1}, \ldots, k^*_{iN}$

Model:
$$\bar{y}_n = \sum_{k=0}^{N} \bar{k}_{ik} u_{n-k}$$

Performance index:
$$Q_N\left(\bar{k}_{i0}, \bar{k}_{i1}, \ldots, \bar{k}_{iN}\right) = \sum_{n=0}^{N} \left(y_n - \bar{y}_n\right)^2 = \sum_{n=0}^{N} \left(y_n - \sum_{k=0}^{n} \bar{k}_{ik} u_{n-k} \right)^2$$

Optimization problem:
$$k^*_{i0}, k^*_{i1}, \ldots, k^*_{iN} \rightarrow Q_N\left(k^*_{i0}, k^*_{i1}, \ldots, k^*_{iN}\right) = \min_{\bar{k}_{i0}, \ldots, \bar{k}_{iN}} Q_N\left(\bar{k}_{i0}, \bar{k}_{i1}, \ldots, \bar{k}_{iN}\right)$$

Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate by the sequence $k_{i0}^*, k_{i1}^*, \ldots, k_{iN}^*$

Solution:

$$\left.\frac{\partial Q_N\left(\bar{k}_{i0}, \bar{k}_{i1}, \ldots, \bar{k}_{iN}\right)}{\partial \bar{k}_{ip}}\right|_{\substack{\bar{k}_{i0}=\bar{k}_{i0}^* \\ \bar{k}_{i1}=\bar{k}_{i1}^* \\ \vdots \\ \bar{k}_{iN}=\bar{k}_{iN}^*}} = 0 \qquad p = 0,1,2,\ldots,N$$

$$-2\sum_{n=0}^{N}\left(y_n - \sum_{k=0}^{n} k_{ik}^* u_{n-k}\right)u_{n-p} = 0 \qquad p = 0,1,2,\ldots,N$$

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Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate by the sequence $k_{i0}^*, k_{i1}^*, \ldots, k_{iN}^*$

$$\sum_{n=0}^{N} y_n u_{n-p} = \sum_{k=0}^{N} \sum_{n=0}^{n} k_{ik}^* u_{n-k} u_{n-p} \qquad p = 0,1,2,\ldots,N$$

Since for $n < 0$ equality $u_n = 0$ holds, we can replace n with N

$$\sum_{n=0}^{N} y_n u_{n-p} = \sum_{k=0}^{N} k_{ik}^* \sum_{n=0}^{N} u_{n-k} u_{n-p} \qquad p = 0,1,2,\ldots,N$$

Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate by the sequence $k^*_{i0}, k^*_{i1}, \ldots, k^*_{iN}$

Let us denote $n-p=\chi$. Then we can rewrite the set of equations:

$$\sum_{\chi=0}^{N-p} y_{\chi+p} u_{\chi} = \sum_{k=0}^{N} k^*_{ik} \sum_{\chi=0}^{N-p} u_{\chi+p-k} u_{\chi} \qquad p=0,1,2,\ldots,N$$

where:

$$\sum_{\chi=0}^{N-p} y_{\chi+p} u_{\chi} = \rho_{yu,p}[p]$$ – correlation of input and output signal

$$\sum_{\chi=0}^{N-p} u_{\chi+p-k} u_{\chi} = \rho_{uu,p}[p-k]$$ – autocorrelation of input signal

Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate by the sequence $k_{i0}^*, k_{i1}^*, \ldots, k_{iN}^*$

The set of equations can be expressed in the equivalent form:

$$\rho_{yu,p}[p] = \sum_{k=0}^{N} k_{ik}^* \rho_{uu,p}[p-k] \qquad p = 0,1,2,\ldots,N$$

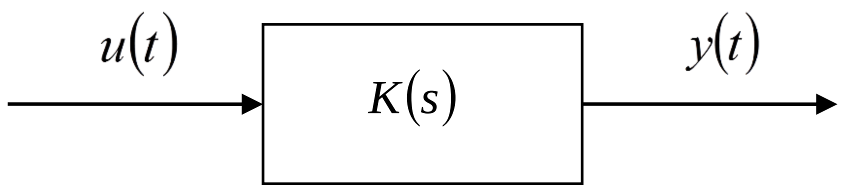
or

$$\begin{bmatrix} \rho_{yu,0}[0] \\ \rho_{yu,1}[1] \\ \vdots \\ \rho_{yu,N}[N] \end{bmatrix} = \begin{bmatrix} \rho_{uu,0}[0] & \rho_{uu,0}[-1] & \cdots & \rho_{uu,0}[-N] \\ \rho_{uu,1}[1] & \rho_{uu,1}[0] & \cdots & \rho_{uu,1}[1-N] \\ \ldots & \ldots & \ldots & \ldots \\ \rho_{uu,N}[N] & \rho_{uu,N}[N-1] & \cdots & \rho_{uu,N}[0] \end{bmatrix} \begin{bmatrix} k_{i0}^* \\ k_{i1}^* \\ \vdots \\ k_{iN}^* \end{bmatrix}$$

$$\overline{\rho}_{yu} = \overline{\rho}_{uu} k_{iN}^* \quad \Downarrow \quad k_{iN}^* = \rho_{uu}^{-1} \rho_{yu}$$

Identification of Dynamic Plants
Identification of Impulse Responses

Continuous case

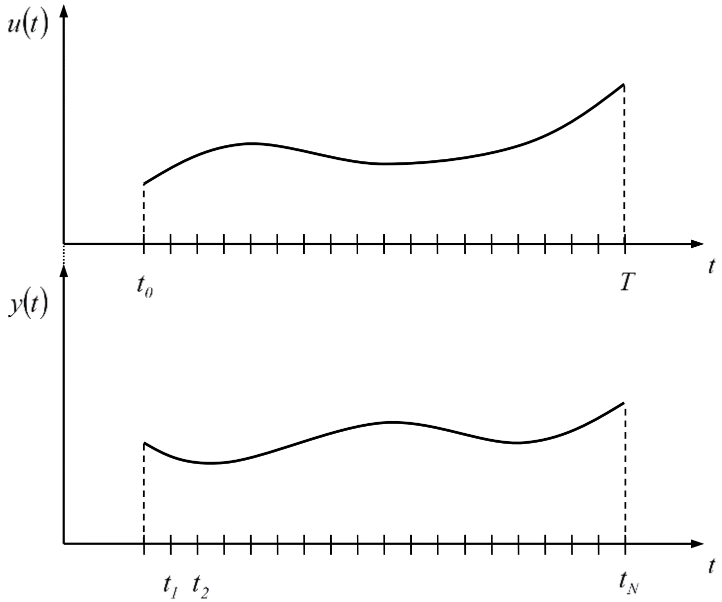


$$k_i(t) = \mathcal{L}^{-1}[K(s)]$$

$$y(t) = \int_0^T k_i(t-\tau)u(\tau)d\tau = \int_0^T k_i(\tau)u(t-\tau)d\tau$$ – continuous case

Identification of Dynamic Plants

For input signal $\{u(t)\}_{t_0}^{T}$ we measure respective output signal $\{y(t)\}_{t_0}^{T}$:



$$t_0 < t_1 < \ldots < t_N \le T$$

$$\{u(t_n)\}_{n=0}^{N} \qquad \{y(t_n)\}_{n=0}^{N}$$

Identification of Dynamic Plants

Identification of Impulse Responses

Continuous Case

We have measurements: $\{u(t)\}_{t=0}^{T}$, $\{y(t)\}_{t=0}^{T}$

and following tasks:

1) plant is linear and we don't know the values of parameters: $k_i(t,\theta)$, θ
2) we want to determine $k_i(t)$

where:

$k_i(t,\theta)$ - known function, θ - unknown vector of parameters

Identification of Dynamic Plants

Identification of Impulse Responses

Continuous Case

3) plant is nonlinear, we approximate: $\bar{k}_i(t,\theta)$, θ

4) plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

where:

$\bar{k}_i(t,\theta)$ - given function, θ - unknown vector of parameters

Identification of Dynamic Plants

Identification of Impulse Responses

ad. 1. plant is linear and we don't know the values of parameters: $k_i(t,\theta)$, θ

$$y(t)=\int_{o}^{t} k_i(\tau,\theta)u(t-\tau)d\tau$$

$$y(t_n)=\int_{o}^{t_n} k_i(\tau,\theta)u(t-\tau)d\tau \qquad n=1,2,\ldots,R$$

Solution of the above system of equations with respect θ gives identification algorithm.

Identification of Dynamic Plants

Identification of Impulse Responses

ad. 2. we want to determine $k_i(t)$

$$k_i(t) = \mathcal{L}^{-1}\left[K(s)\right]$$

$$K(s) = \frac{\mathcal{L}\left[y(t)\right]}{\mathcal{L}\left[u(t)\right]}$$

Identification of Dynamic Plants

Identification of Impulse Responses

ad. 3. Plant is nonlinear, we approximate: $\bar{k}_i(t,\theta)$, θ

where:

Model:

$$\bar{y}(t,\theta)=\int_o^t \bar{k}_i(\tau,\theta)u(t-\tau)d\tau$$

Approximation:

Performance index: $$Q_T(\theta)=\int_0^T (y(t)-\bar{y}(t,\theta))^2 dt=\int_0^T \left(y(t)-\int_o^t k_i(\tau,\theta)u(t-\tau)d\tau \right)^2 dt$$

Optimization problem: $$\theta_T^* \to Q_T(\theta^*)=\min_\theta Q_T(\theta)$$

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Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

Model:

$$\bar{y}(t) = \int_o^t \bar{k}_i(\tau) u(t-\tau) d\tau$$

Performance index:

$$Q(\bar{k}_i(\tau)) = \int_0^T (y(t) - \bar{y}(t))^2 dt = \int_0^T \left(y(t) - \int_o^t \bar{k}_i(\tau) u(t-\tau) d\tau \right)^2 dt$$

Optimization task (minimization of functional):

$$k_i^*(\tau) \rightarrow Q_T(k_i^*(\tau)) = \min_{\bar{k}_i(\tau)} Q_T(\bar{k}_i(\tau))$$

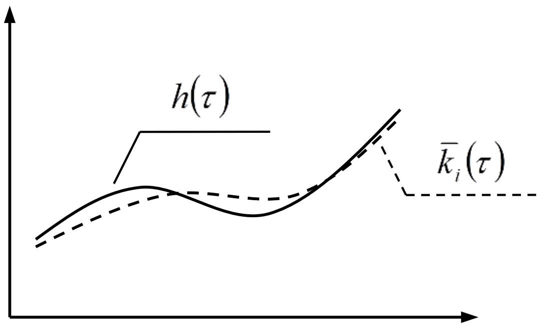
Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

$$k_i^*(\tau) \rightarrow Q_T\left(k_i^*(\tau)\right) = \min_{\bar{k}_i(\tau)} Q_T\left(\bar{k}_i(\tau)\right)$$

$$\bar{k}_i(\tau) + \delta h(\tau)$$

$$\left.\frac{\partial Q_T}{\partial \delta}\right|_{\delta=0} = 0, \qquad \forall h(\tau)$$



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

$$Q\left(\bar{k}_i(\tau)+\delta h(\tau)\right)=\int_0^T\left(y(t)-\int_0^t\left(\bar{k}_i(\tau)+\delta h(\tau)\right)u(t-\tau)d\tau\right)^2 dt=$$

$$=\int_0^T\left(y(t)-\int_0^t\bar{k}_i(\tau)u(t-\tau)d\tau-\delta\int_0^t h(\tau)u(t-\tau)d\tau\right)^2 dt=\int_o^T\left(y(t)-\int_0^t\bar{k}_i(\tau)u(t-\tau)d\tau\right)^2 dt+$$

$$-2\delta\int_0^T\left(y(t)-\int_0^t\bar{k}_i(\tau)u(t-\tau)d\tau\right)\int_0^t h(\chi)u(t-\chi)d\chi dt+\delta^2\int_0^T\left(\int_0^t h(\tau)u(t-\tau)d\tau\right)^2 dt=$$

$$=Q_1-2\delta Q_2+\delta^2 Q_3=Q$$

$$\frac{\partial Q}{\partial\delta}=-2Q_2+2\delta Q_3\Big|_{\delta=0}=0\ \Rightarrow\ Q_2=0\quad\forall h(\tau)$$

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Identification of Dynamic Plants
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

$$\forall h(\tau) \quad \int_0^T \left(y(t) - \int_0^t k_i^*(\tau) u(t-\tau) d\tau \right) \int_0^t h(\chi) u(t-\chi) d\chi dt = 0$$

$u(t)=0$ for $t<0$, so $t \to T$

$$\forall h(\tau) \quad \int_0^T \left(y(t) - \int_0^T k_i^*(\tau) u(t-\tau) d\tau \right) \int_0^T h(\chi) u(t-\chi) d\chi dt = 0$$

Identification of Dynamic Plants

Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

$$\forall h(\tau) \quad \int_0^T h(\chi) \underbrace{\left(y(t)u(t-\chi)dt - \int_0^T k_i^*(\tau) \int_o^T u(t-\tau)u(t-\chi)dtd\tau \right)}_{\Downarrow \atop 0} d\chi = 0$$

$$\downarrow$$

$$\int_o^T y(t)u(t-\chi)dt = \int_0^T k_i^*(\tau) \int_o^T u(t-\tau)u(t-\chi)dtd\tau$$

where: $t-\chi=\gamma$

Identification of Dynamic Plants

Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

$$\int_o^{T-\gamma} y(\gamma+\chi)u(\gamma)dt = \int_0^T k_i^*(\tau)\int_o^{T-\gamma} u(\gamma)u(\gamma+\chi-\tau)dtd\tau$$

$$\rho_{yu}(\chi) = \int_0^T k_i^*(\tau)\rho_{uu}(\chi-\tau)d\tau$$

Thank you for attention