

Computer Science

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Systems Modelling and Analysis

Choose yourself and new technologies

L.14. Identification of dynamic plants



HUMAN CAPITAL
HUMAN – BEST INVESTMENT!



Wrocław University of Technology

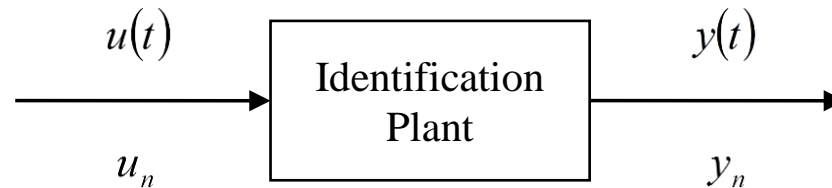
EUROPEAN
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Identification of Dynamic Plants



Descriptions:

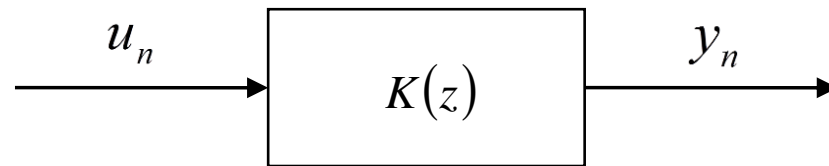
- state variable;
- differential/difference equation;
- transfer functions: $K(s)$, $K(z)$;
- impulse response: $k_i(t)$, k_{in} ;
- step response: $h(t)$, $h_n(t)$.



Identification of Dynamic Plants

Identification of Impulse Responses

Discrete case



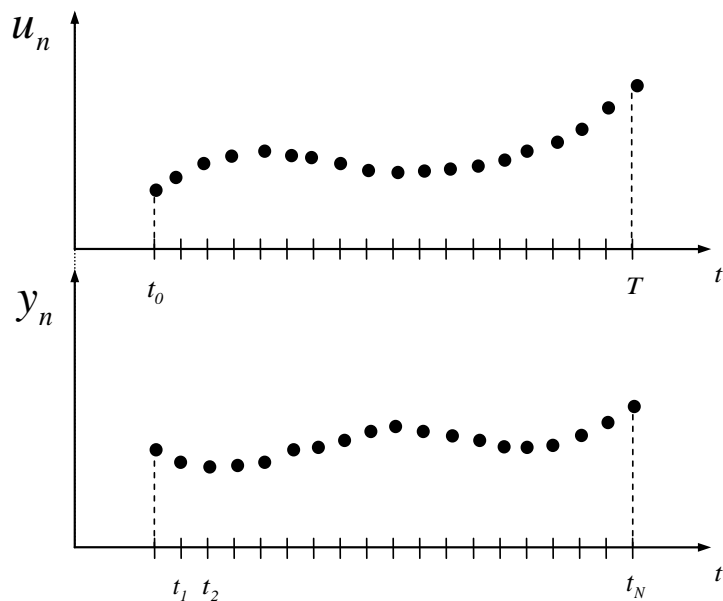
$$k_{in} = \mathcal{Z}^{-1}[K(z)]$$

$$y_n = \sum_{k=0}^n k_{ik} u_{n-k} = \sum_{k=0}^n k_{in-k} u_k \quad \text{– discrete case}$$



Identification of Dynamic Plants

For input signal u_n we measure respective output signal y_n :



$$t_0 < t_1 < \dots < t_N \leq T$$

$$\{u_n\}_{n=0}^N \quad \{y_n\}_{n=0}^N$$



Identification of Dynamic Plants

Identification of Impulse Responses

Discrete case

We want to determine discrete input response (for $u_n = \delta_n$): k_{in}

We have sequence of measurements: u_0, u_1, \dots, u_N
 y_0, y_1, \dots, y_N

and following tasks:

- 1) plant is linear: $k_{in}(\theta)$, but we don't know the values of parameters θ
- 2) we want to determine sequence of values of $k_{i0}, k_{i1}, \dots, k_{iN}$

where:

k_{in} - impulse function



Identification of Dynamic Plants

Identification of Impulse Responses

Discrete case

- 3) plant is nonlinear, we approximate: $\bar{k}_{in}(\theta), \theta^*$
- 4) plant is nonlinear, we approximate by the sequence of discrete impulse response $k_{i0}^*, k_{i1}^*, \dots, k_{iN}^*$

where:

\bar{k}_{in} - given function

θ - unknown vector of parameter



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 1. Linear plant

$$y_n = \sum_{k=0}^N k_{ik}(\theta) u_{n-k}$$

where:

k_{ik} - given function, θ - unknown

$$y_0 = k_{i0}(\theta) u_0$$

$$y_1 = k_{i0}(\theta) u_1 + k_{i1}(\theta) u_0$$

\vdots

$$y_N = k_{i0}(\theta) u_N + k_{i1}(\theta) u_{N-1} + \dots + k_{iN}(\theta) u_0$$

$$N \cdot L \geq R$$

$$\dim y = L$$

$$\dim \theta = R$$

Solution of the above system of equations is calculated values of θ



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 2. Determination of discrete impulse responses values $k_{i0}, k_{i1}, \dots, k_{iN}$

$$y_n = \sum_{k=0}^n k_{ik} u_{n-k}$$

$$\left\{ \begin{array}{l} y_0 = k_{i0} u_0 \\ y_1 = k_{i0} u_1 + k_{i1} u_0 \\ \vdots \\ y_N = k_{i0} u_N + k_{i1} u_{N-1} + \dots + k_{iN} u_0 \end{array} \right.$$



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 2. Determination of discrete impulse responses values $k_{i0}, k_{i1}, \dots, k_{iN}$

Denote:

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix} = Y_N \quad \begin{bmatrix} u_0 & 0 & \dots & 0 \\ u_1 & u_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ u_N & u_{N-1} & \dots & u_0 \end{bmatrix} = U_N \quad \begin{bmatrix} k_{i0} \\ k_{i1} \\ \vdots \\ k_{iN} \end{bmatrix} = \mathbf{k}_{iN}$$

Solution:

$$Y_N = U_N \mathbf{k}_{in} \Rightarrow \underline{\underline{\mathbf{k}_{in} = U_N^{-1} Y_N}}$$



Identification of Dynamic Plants

Identification of Impulse Responses

Ad. 3. Plant is nonlinear; approximation by $\bar{k}_{in}(\theta)$.

$$\bar{y}_n = \sum_{k=0}^N \bar{k}_{ik}(\theta) u_{n-k} \quad - \text{approximation}$$

where: \bar{k}_{ik} - given function, θ - unknown vector of parameters

Performance index:

$$Q_N(\theta) = \sum_{n=0}^N (y_n - \bar{y}_n)^2 = \sum_{n=0}^N \left(y_n - \sum_{k=0}^n \bar{k}_{ik}(\theta) u_{n-k} \right)^2$$

Optimization problem:

$$\theta_N^* \rightarrow Q_N(\theta_N^*) = \min_{\theta \in \Theta} Q_N(\theta)$$



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate by the sequence of discrete impulse responses $k_{i0}^*, k_{i1}^*, \dots, k_{iN}^*$

Model:

$$\bar{y}_n = \sum_{k=0}^N \bar{k}_{ik} u_{n-k}$$

Performance index:

$$Q_N(\bar{k}_{i0}, \bar{k}_{i1}, \dots, \bar{k}_{iN}) = \sum_{n=0}^N (y_n - \bar{y}_n)^2 = \sum_{n=0}^N \left(y_n - \sum_{k=0}^n \bar{k}_{ik} u_{n-k} \right)^2$$

Optimization problem:

$$k_{i0}^*, k_{i1}^*, \dots, k_{iN}^* \rightarrow Q_N(k_{i0}^*, k_{i1}^*, \dots, k_{iN}^*) = \min_{\bar{k}_{i0}, \dots, \bar{k}_{iN}} Q_N(\bar{k}_{i0}, \bar{k}_{i1}, \dots, \bar{k}_{iN})$$



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate by the sequence $k_{i0}^*, k_{i1}^*, \dots, k_{iN}^*$

Solution:

$$\frac{\partial Q_N(\bar{k}_{i0}, \bar{k}_{i1}, \dots, \bar{k}_{iN})}{\partial \bar{k}_{ip}} \bigg|_{\substack{\bar{k}_{i0} = \bar{k}_{i0}^* \\ \bar{k}_{i1} = \bar{k}_{i1}^* \\ \vdots \\ \bar{k}_{iN} = \bar{k}_{iN}^*}} = 0 \quad p = 0, 1, 2, \dots, N$$

$$-2 \sum_{n=0}^N \left(y_n - \sum_{k=0}^n k_{ik}^* u_{n-k} \right) u_{n-p} = 0 \quad p = 0, 1, 2, \dots, N$$



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate by the sequence $k_{i0}^*, k_{i1}^*, \dots, k_{iN}^*$

$$\sum_{n=0}^N y_n u_{n-p} = \sum_{k=0}^N \sum_{n=0}^k k_{ik}^* u_{n-k} u_{n-p} \quad p = 0, 1, 2, \dots, N$$

Since for $n < 0$ equality $u_n = 0$ holds, we can replace n with N

$$\sum_{n=0}^N y_n u_{n-p} = \sum_{k=0}^N k_{ik}^* \sum_{n=0}^N u_{n-k} u_{n-p} \quad p = 0, 1, 2, \dots, N$$



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate by the sequence $k_{i0}^*, k_{i1}^*, \dots, k_{iN}^*$

Let us denote $n - p = \chi$. Then we can rewrite the set of equations:

$$\sum_{\chi=0}^{N-p} y_{\chi+p} u_{\chi} = \sum_{k=0}^N k_{ik}^* \sum_{\chi=0}^{N-p} u_{\chi+p-k} u_{\chi} \quad p = 0, 1, 2, \dots, N$$

where:

$$\sum_{\chi=0}^{N-p} y_{\chi+p} u_{\chi} = \rho_{yu,p}[p] \quad \text{– correlation of input and output signal}$$

$$\sum_{\chi=0}^{N-p} u_{\chi+p-k} u_{\chi} = \rho_{uu,p}[p-k] \quad \text{– autocorrelation of input signal}$$



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate by the sequence $k_{i0}^*, k_{i1}^*, \dots, k_{iN}^*$

The set of equations can be expressed in the equivalent form:

$$\rho_{yu,p}[p] = \sum_{k=0}^N k_{ik}^* \rho_{uu,p}[p-k] \quad p = 0, 1, 2, \dots, N$$

or

$$\begin{bmatrix} \rho_{yu,0}[0] \\ \rho_{yu,1}[1] \\ \vdots \\ \rho_{yu,N}[N] \end{bmatrix} = \begin{bmatrix} \rho_{uu,0}[0] & \rho_{uu,0}[-1] & \cdots & \rho_{uu,0}[-N] \\ \rho_{uu,1}[1] & \rho_{uu,1}[0] & \cdots & \rho_{uu,1}[1-N] \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{uu,N}[N] & \rho_{uu,N}[N-1] & \cdots & \rho_{uu,N}[0] \end{bmatrix} \begin{bmatrix} k_{i0}^* \\ k_{i1}^* \\ \vdots \\ k_{iN}^* \end{bmatrix}$$

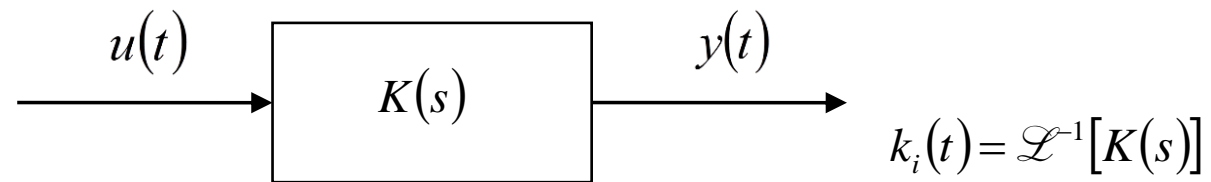
$$\begin{aligned} \bar{\rho}_{yu} &= \bar{\rho}_{uu} k_{iN}^* \\ &\Downarrow \\ \underline{\underline{k_{iN}^*}} &= \underline{\underline{\rho_{uu}^{-1} \rho_{yu}}} \end{aligned}$$



Identification of Dynamic Plants

Identification of Impulse Responses

Continuous case

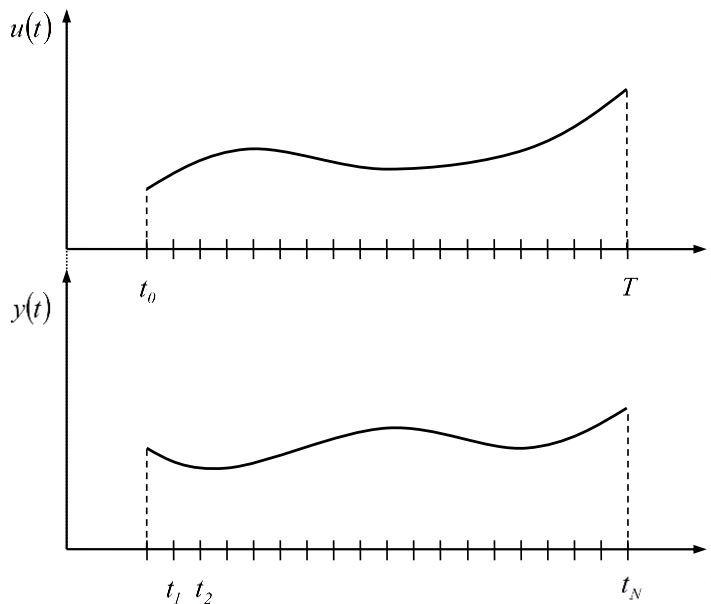


$$y(t) = \int_0^T k_i(t - \tau) u(\tau) d\tau = \int_0^T k_i(\tau) u(t - \tau) d\tau \quad \text{-- continuous case}$$



Identification of Dynamic Plants

For input signal $\{u(t)\}_{t_0}^T$ we measure respective output signal $\{y(t)\}_{t_0}^T$:



$$t_0 < t_1 < \dots < t_N \leq T$$

$$\{u(t_n)\}_{n=0}^N \quad \{y(t_n)\}_{n=0}^N$$



Identification of Dynamic Plants

Identification of Impulse Responses

Continuous Case

We have measurements: $\{u(t)\}_{t=0}^T, \{y(t)\}_{t=0}^T$

and following tasks:

- 1) plant is linear and we don't know the values of parameters: $k_i(t, \theta), \theta$
- 2) we want to determine $k_i(t)$

where:

$k_i(t, \theta)$ - known function, θ - unknown vector of parameters



Identification of Dynamic Plants

Identification of Impulse Responses

Continuous Case

3) plant is nonlinear, we approximate: $\bar{k}_i(t, \theta)$, θ

4) plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

where:

$\bar{k}_i(t, \theta)$ - given function, θ - unknown vector of parameters



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 1. plant is linear and we don't know the values of parameters: $k_i(t, \theta)$, θ

$$y(t) = \int_0^t k_i(\tau, \theta) u(t - \tau) d\tau$$

$$y(t_n) = \int_0^{t_n} k_i(\tau, \theta) u(t - \tau) d\tau \quad n = 1, 2, \dots, R$$

Solution of the above system of equations with respect θ gives identification algorithm.



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 2. we want to determine $k_i(t)$

$$k_i(t) = \mathcal{L}^{-1}[K(s)]$$

$$K(s) = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[u(t)]}$$



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 3. Plant is nonlinear, we approximate: $\bar{k}_i(t, \theta), \theta$

where:

Model:

$$\bar{y}(t, \theta) = \int_0^t \bar{k}_i(\tau, \theta) u(t - \tau) d\tau$$

Approximation:

Performance index:
$$Q_T(\theta) = \int_0^T (y(t) - \bar{y}(t, \theta))^2 dt = \int_0^T \left(y(t) - \int_0^t k_i(\tau, \theta) u(t - \tau) d\tau \right)^2 dt$$

Optimization problem:

$$\theta_T^* \rightarrow Q_T(\theta^*) = \min_{\theta} Q_T(\theta)$$



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

Model:

$$\bar{y}(t) = \int_0^t \bar{k}_i(\tau) u(t-\tau) d\tau$$

Performance index:

$$Q(\bar{k}_i(\tau)) = \int_0^T (y(t) - \bar{y}(t))^2 dt = \int_0^T \left(y(t) - \int_0^t \bar{k}_i(\tau) u(t-\tau) d\tau \right)^2 dt$$

Optimization task (minimization of functional):

$$k_i^*(\tau) \rightarrow Q_T(k_i^*(\tau)) = \min_{\bar{k}_i(\tau)} Q_T(\bar{k}_i(\tau))$$



Identification of Dynamic Plants

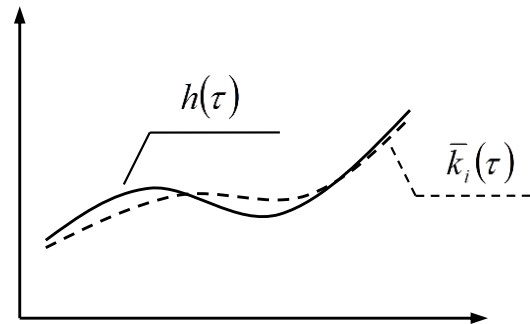
Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

$$k_i^*(\tau) \rightarrow Q_T(k_i^*(\tau)) = \min_{\bar{k}_i(\tau)} Q_T(\bar{k}_i(\tau))$$

$$\bar{k}_i(\tau) + \delta h(\tau)$$

$$\left. \frac{\partial Q_T}{\partial \delta} \right|_{\delta=0} = 0, \quad \forall h(\tau)$$





Identification of Dynamic Plants

Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

$$\begin{aligned}
 Q(\bar{k}_i(\tau) + \delta h(\tau)) &= \int_0^T \left(y(t) - \int_0^t (\bar{k}_i(\tau) + \delta h(\tau)) u(t-\tau) d\tau \right)^2 dt = \\
 &= \int_0^T \left(y(t) - \int_0^t \bar{k}_i(\tau) u(t-\tau) d\tau - \delta \int_0^t h(\tau) u(t-\tau) d\tau \right)^2 dt = \int_0^T \left(y(t) - \int_0^t \bar{k}_i(\tau) u(t-\tau) d\tau \right)^2 dt + \\
 &- 2\delta \int_0^T \left(y(t) - \int_0^t \bar{k}_i(\tau) u(t-\tau) d\tau \right) \int_0^t h(\chi) u(t-\chi) d\chi dt + \delta^2 \int_0^T \left(\int_0^t h(\tau) u(t-\tau) d\tau \right)^2 dt = \\
 &= Q_1 - 2\delta Q_2 + \delta^2 Q_3 = Q
 \end{aligned}$$

$$\frac{\partial Q}{\partial \delta} = -2Q_2 + 2\delta Q_3 \Big|_{\delta=0} = 0 \Rightarrow Q_2 = 0 \quad \forall h(\tau)$$



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

$$\forall h(\tau) \quad \int_0^T \left(y(t) - \int_0^t k_i^*(\tau) u(t-\tau) d\tau \right) \int_0^t h(\chi) u(t-\chi) d\chi dt = 0$$

$u(t)=0$ for $t < 0$, so $t \rightarrow T$

$$\forall h(\tau) \quad \int_0^T \left(y(t) - \int_0^T k_i^*(\tau) u(t-\tau) d\tau \right) \int_0^T h(\chi) u(t-\chi) d\chi dt = 0$$



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

$$\forall h(\tau) \quad \int_0^T h(\chi) \left(\underbrace{y(t)u(t-\chi)dt - \int_0^T k_i^*(\tau) \int_0^T u(t-\tau)u(t-\chi)dtd\tau}_{\downarrow 0} \right) d\chi = 0$$

$$\downarrow$$

$$\int_0^T y(t)u(t-\chi)dt = \int_0^T k_i^*(\tau) \int_0^T u(t-\tau)u(t-\chi)dtd\tau$$

where: $t - \chi = \gamma$



Identification of Dynamic Plants

Identification of Impulse Responses

ad. 4. Plant is nonlinear, we approximate impulse response by $\bar{k}_i(t)$

$$\int_0^{T-\gamma} y(\gamma + \chi) u(\gamma) dt = \int_0^T k_i^*(\tau) \int_0^{T-\gamma} u(\gamma) u(\gamma + \chi - \tau) dt d\tau$$

$$\rho_{yu}(\chi) = \int_0^T k_i^*(\tau) \rho_{uu}(\chi - \tau) d\tau$$



Thank you for attention

