

Computer Science

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Systems Modelling and Analysis

Choose yourself and new technologies

L.12. Multicriteria decision making



HUMAN CAPITAL
HUMAN – BEST INVESTMENT!



Wrocław University of Technology

EUROPEAN
SOCIAL FUND



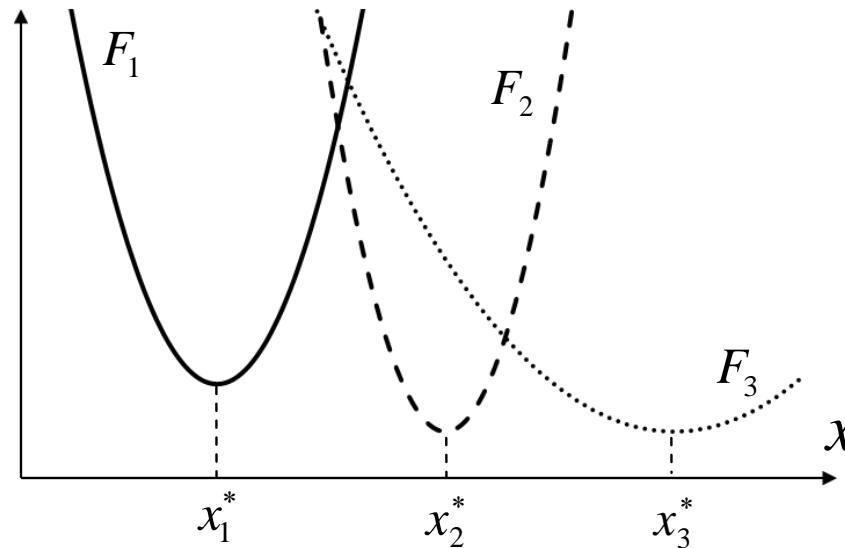
Project co-financed from the EU European Social Fund



Multicriteria optimization

x – decision variables vector

$F_1(x), F_2(x), \dots, F_K(x)$ – performance indices





Multicriteria optimization

Synthetic performance index

$$F(x) = H(F_1(x), F_2(x), \dots, F_K(x))$$

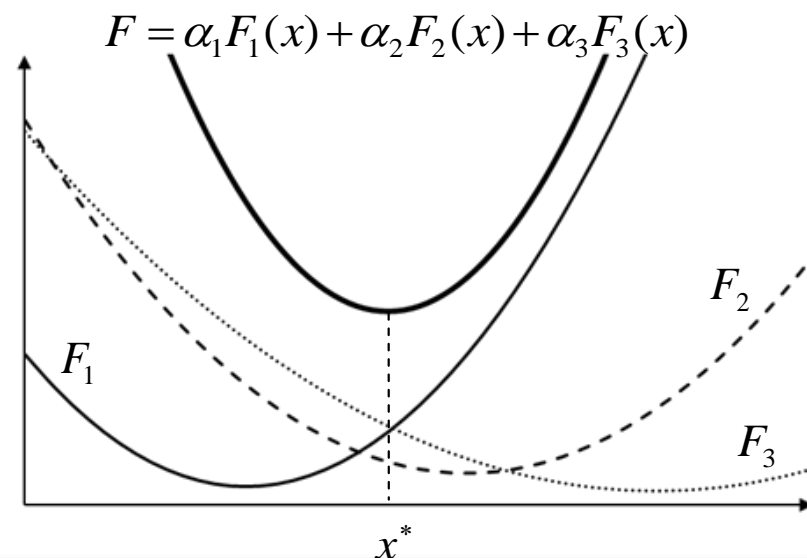
e. g.:
$$F(x) = \sum_{k=1}^K \alpha_k F_k(x)$$

where:
$$\sum_{k=1}^K \alpha_k = 1, \alpha_k > 0, k = 1, 2, \dots, K$$

$$F(x) = \prod_{k=1}^K F_k(x)$$

$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$H(\cdot)$ – monotonic for all variables





Multicriteria optimization

A selected performance index is optimized,

Upper limits for values of another performance indices are specified.

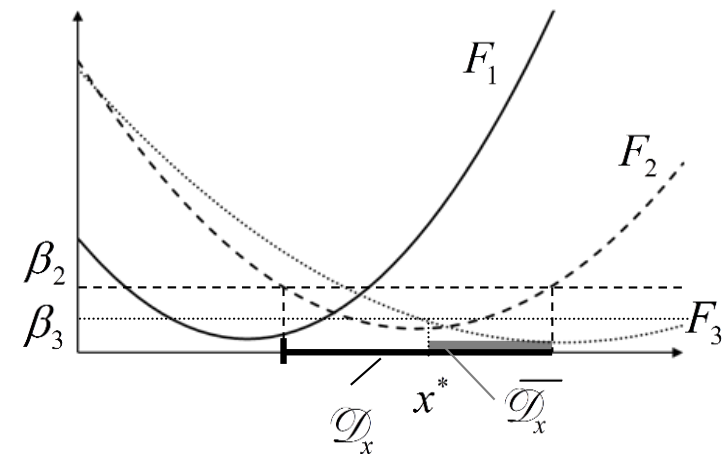
Let $F_1(x)$ be a selected performance index

$$F_k(x) \leq \beta_k, \quad k = 2, 3, \dots, K$$

Requirements for performance indices are met

$$\overline{\mathcal{D}}_x = \mathcal{D}_x \cap \{x \in \mathcal{R}^s : F_k(x) \leq \beta_k, k = 2, \dots, K\}$$

$$x^* \rightarrow F_1(x^*) = \min_{x \in \overline{\mathcal{D}}_x} F_1(x)$$





Multicriteria optimization

Ranked/prioritized performance indices

$$F_1(x) \succ F_2(x) \succ \dots \succ F_K(x) \quad x \in \mathcal{D}_x$$

Step 1. $\mathcal{D}_{x_1} = \mathcal{D}_x$

$$x_1^* \rightarrow F_1(x_1^*) = \min_{x \in \mathcal{D}_{x_1}} F_1(x)$$

Step 2. $\mathcal{D}_{x_2} = \mathcal{D}_{x_1} \cap \left\{ x \in \mathcal{R}^S : F_1(x) \leq F_1(x_1^*) + \gamma_1 \right\}$

$$x_2^* \rightarrow F_2(x_2^*) = \min_{x \in \mathcal{D}_{x_2}} F_2(x)$$

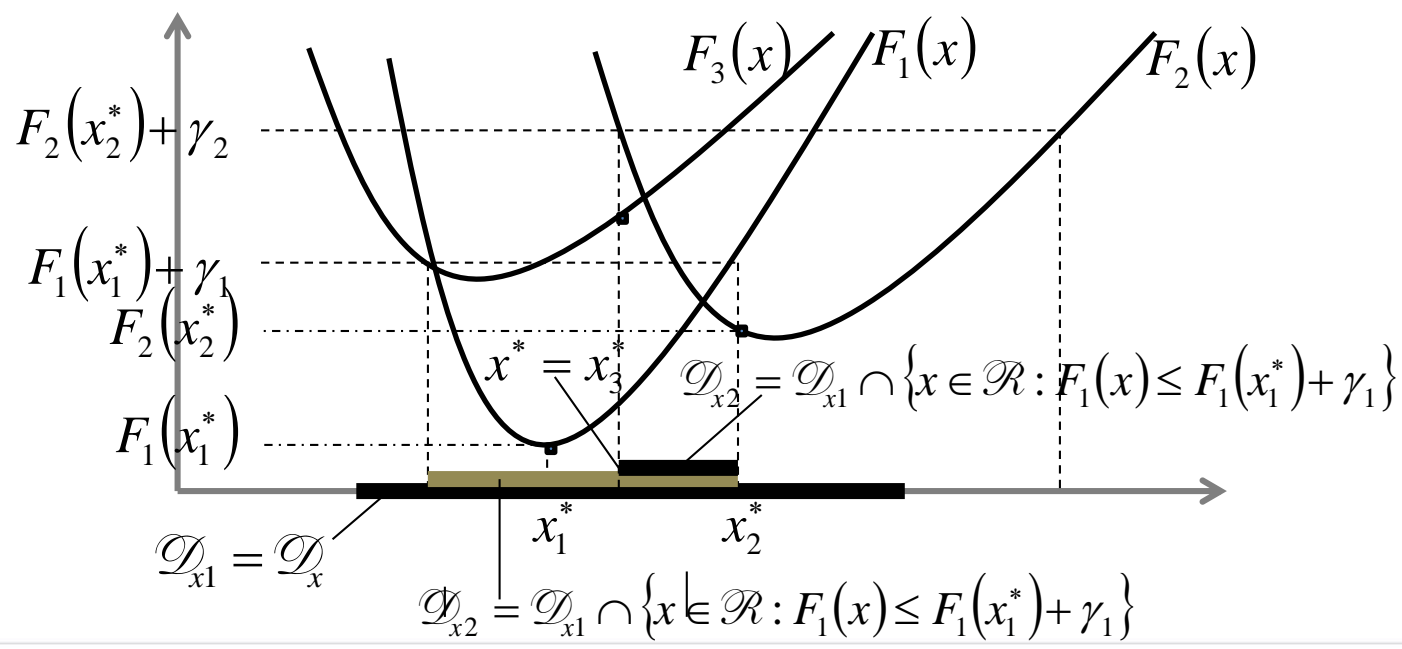
⋮



Multicriteria optimization

Step K. $\mathcal{D}_{xK} = \mathcal{D}_{xK-1} \cap \{x \in \mathcal{R}^S : F_{K-1}(x) \leq F_1(x_{K-1}^*) + \gamma_{K-1}\}$

$x_K^* = x_K^* \rightarrow F_K(x_K^*) = \min_{x \in \mathcal{D}_{xK}} F_K(x)$





Non-dominated solutions

Pareto-optimality

$$F_1(x), F_2(x), \dots, F_K(x) \quad x \in \mathcal{D}_x$$

D_K – a set of non-dominated solutions

A set of non-dominated solutions $x \in \mathcal{D}_x$ is a subset of such points $D_K \subseteq \mathcal{D}_x$ for which performance indices has the following property: any of performance indices may decrease only if at least one of the remaining performance indices increases.

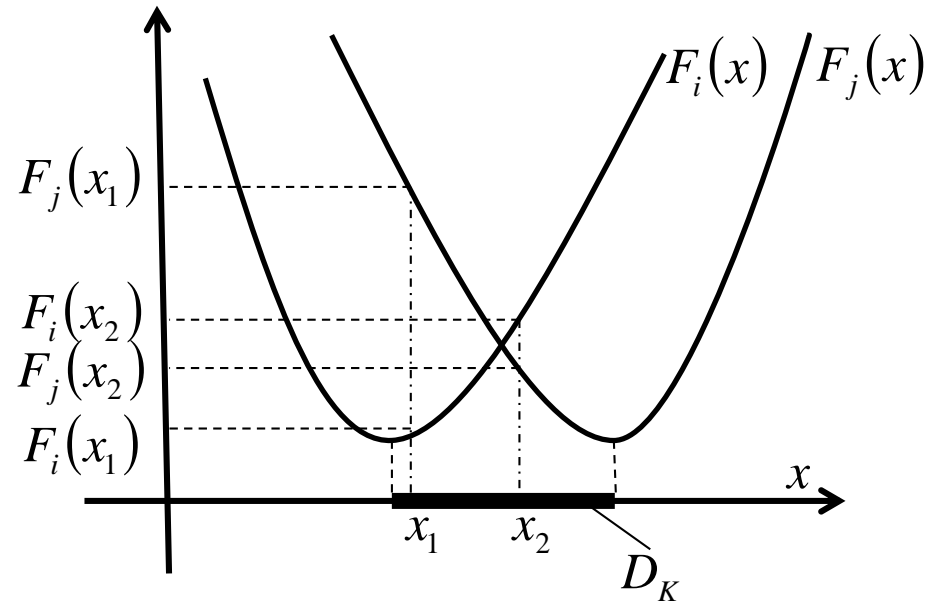
$$x_1, x_2 \in D_K \Leftrightarrow \forall i \in \{1, 2, \dots, K\} \exists j \in \{1, 2, \dots, K\} \\ F_i(x_1) < F_j(x_2) \Rightarrow F_j(x_1) < F_i(x_2)$$



Non-dominated solutions

$$x_1, x_2 \in D_K \Leftrightarrow \forall j \in \{1, 2, \dots, K\} \exists i \in \{1, 2, \dots, K\}$$

$$F_j(x_1) > F_j(x_2) \Rightarrow F_i(x_1) < F_i(x_2)$$





Non-dominated solutions

Analytical condition
$$\sum_{k=1}^K \eta_k \nabla_x F_k(x) = 0_S \quad \eta_k \geq 0 \quad k = 1, 2, \dots, K$$

Example:

$$F_1(x) = (x^{(1)} - 1)^2 + (x^{(2)} - 1)^2, \quad F_2(x) = (x^{(1)} + 1)^2 + (x^{(2)} + 1)^2$$

$$\eta_1 \begin{bmatrix} 2(x^{(1)} - 1) \\ 2(x^{(2)} - 1) \end{bmatrix} + \eta_2 \begin{bmatrix} 2(x^{(1)} + 1) \\ 2(x^{(2)} + 1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad /2 \quad \eta_1, \eta_2 \geq 0$$

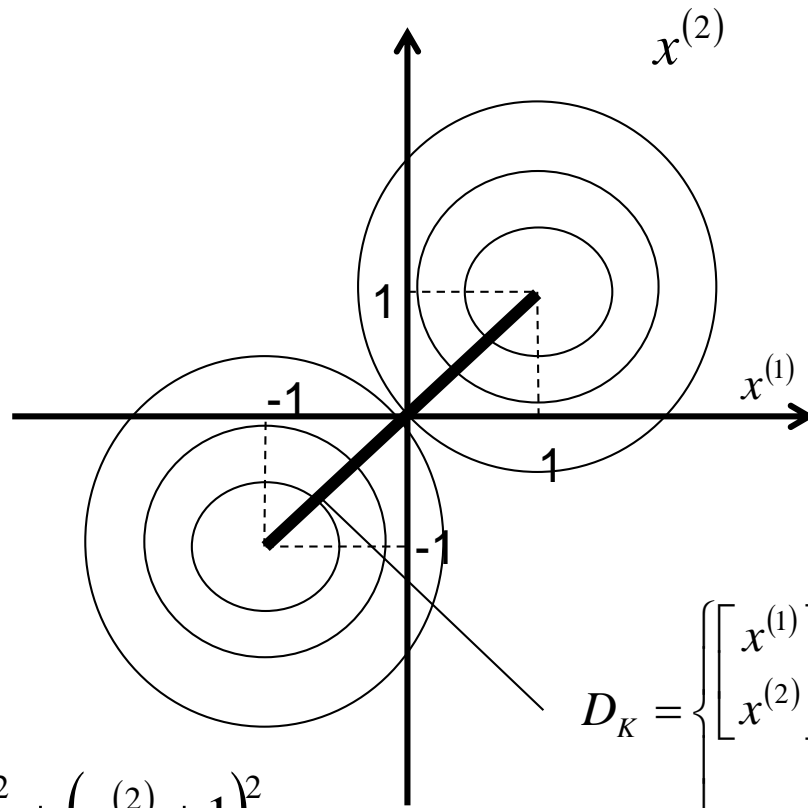
$$\eta_1 \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} + \eta_2 \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} = \eta_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \eta_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad /(\eta_1 + \eta_2)$$

$$\frac{\eta_1}{\eta_1 + \eta_2} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} + \frac{\eta_2}{\eta_1 + \eta_2} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} = \frac{\eta_1}{\eta_1 + \eta_2} \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \frac{\eta_2}{\eta_1 + \eta_2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} + = \lambda_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ gdzie } \lambda_1 = \frac{\eta_1}{\eta_1 + \eta_2}, \lambda_2 = \frac{\eta_2}{\eta_1 + \eta_2}, \lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 \in [0, 1]$$



Non-dominated solutions



$$F_1(x) = (x^{(1)} - 1)^2 + (x^{(2)} - 1)^2$$

$$F_2(x) = (x^{(1)} + 1)^2 + (x^{(2)} + 1)^2$$

$$D_K = \left\{ \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} \in \mathbb{R}^2 : \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} + = \lambda_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \right. \\ \left. \lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 \in [0, 1] \right\}$$



Thank you for attention

