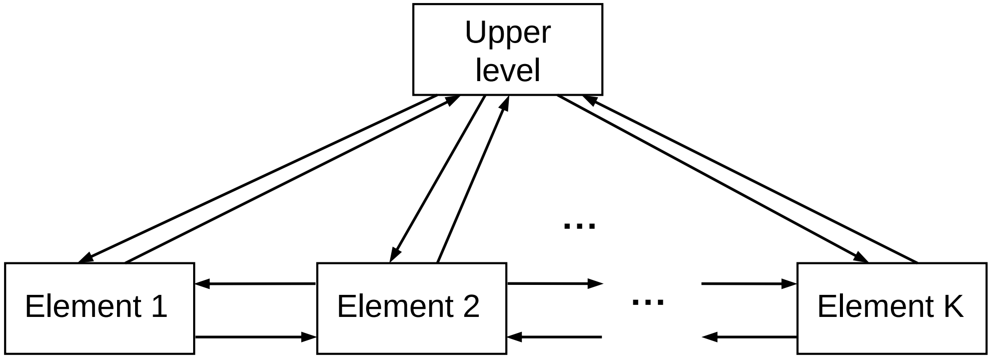
Politechnika Wrocławska

Decomposition and coordination of the optimization task

Decomposition of the optimization task - motivation

$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

1. Large dimension of the decision vector - decomposition of the optimization task.
2. Subsets of the decisions variables are connected with quite different problems (elements).
3. Complex structure of the decision problem – particular decisions are connected with quite different plants, for example two stage management system,



Separowalna funkcja celu

$$x^* \to F(x^*) = \min_{x \in \mathscr{R}^S} F(x)$$

$$x = \begin{bmatrix} x^I \\ x^{II} \\ \vdots \\ x^K \end{bmatrix}, x \in \mathscr{R}^S = \mathscr{R}^{S_I} \times \mathscr{R}^{S_{II}} \times \cdots \times \mathscr{R}^{S_K}, x^k \in \mathscr{R}^{S_k}, k = I, II, \ldots, K, \sum_{k=I}^{K} S_k = S$$

$$F(x) = H\left(F_I(x^I), F_{II}(x^{II}), \ldots, F_K(x^K)\right)$$

Function $H(.)$ is monotonic function with respect of all variable - examples:

$$F(x) = \sum_{k=I}^{K} \alpha_k F_k\left(x^k\right), \alpha_k \geq 0, k = I, II, \ldots, K$$

$$F(x) = \prod_{k=I}^{K} F_k\left(x^k\right)$$

Separable goal function

$$\min_{x\in\mathcal{R}^{\mathcal{S}}} F(x) = \min_{x\in\mathcal{R}^{\mathcal{S}}} H\left(F_I(x^I), F_{II}(x^{II}), \ldots, F_K(x^K)\right) =$$

$$H\left(\min_{x^I\in\mathcal{R}^{\mathcal{S}_I}} F_I(x^I), \min_{x^{II}\in\mathcal{R}^{\mathcal{S}_{II}}} F_{II}(x^{II}), \ldots, \min_{x^K\in\mathcal{R}^{\mathcal{S}_K}} F_K(x^K)\right)$$

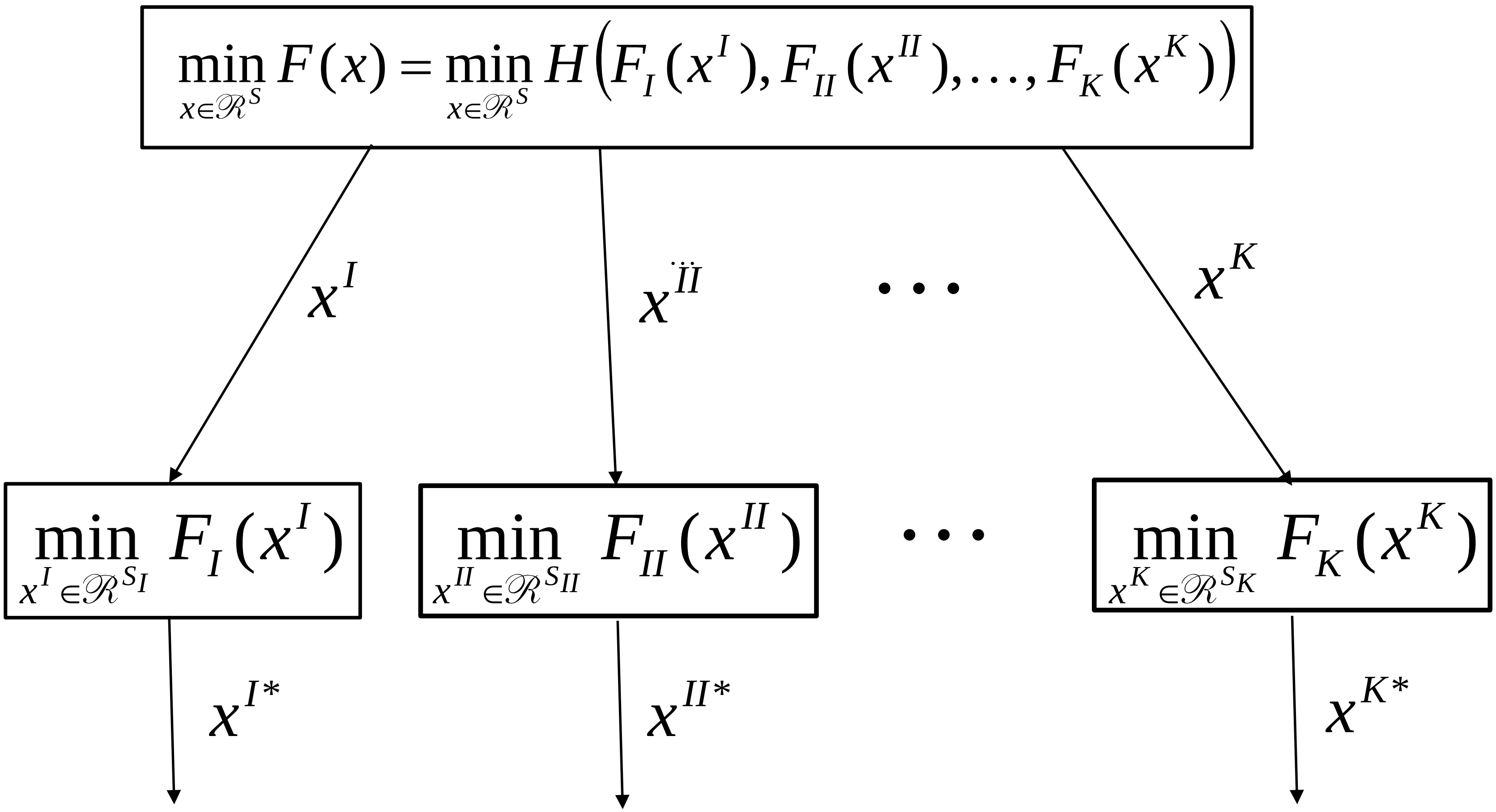
In these case the complex optimization problem may be decomposed on to K independent smaller optimization tasks, i.e.: optimisation task

$$x^* \rightarrow F(x^*) = \min_{x\in\mathcal{R}^S} F(x)$$

is equivalent:

$$x^{k*} \rightarrow F_k(x^{k*}) = \min_{x^k\in\mathcal{R}^{S_k}} F_k(x^k), k = I, II, \ldots, K$$

Separable goal function



Separable goal function - example

$$F(x) = x^T A x + b^T x$$

$$x = \begin{bmatrix} x^I \\ x^{II} \\ \vdots \\ x^K \end{bmatrix}, b = \begin{bmatrix} b^I \\ b^{II} \\ \vdots \\ b^K \end{bmatrix}, A = \begin{bmatrix} A_I & 0 & \cdots & 0 \\ 0 & A_{II} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A_K \end{bmatrix}, \begin{matrix} x, b \in R^S = R^{S_I} \times R^{S_{II}} \times \cdots \times R^{S_K}, \\ x^k, b^k \in R^{S_k}, A_k \in R^{S_k} \times R^{S_k}, \\ k = I, II, \ldots, K, \sum_{k=I}^{K} S_k = S \end{matrix}$$

$$F(x) = x^T A x + b^T x =$$

$$= {x^I}^T A_I x^I + {b^I}^T x^I + {x^{II}}^T A_{II} x^{II} + {b^{II}}^T x_{II} + \cdots + {x^K}^T A_K x^K + {b^K}^T x^K$$

Separable goal function - example

$$x^* \to F(x^*) = \min_{x \in \mathcal{R}^{S}} F(x) = \min_{x \in \mathcal{R}^{S}} \left(x^T A x + b^T x \right) =$$

$$= \min_{x \in \mathcal{R}^{S_I}} \left(x^{I^T} A_I x^I + b^{I^T} x^I \right) + \min_{x \in \mathcal{R}^{S_{II}}} \left(x^{II^T} A_{II} x^{II} + b^{II^T} x_{II} \right) + \cdots + \min_{x \in \mathcal{R}^{S_K}} \left(x^{K^T} A_K x^K + b^{K^T} x^K \right)$$

$$x^{k*} \to x^{k*T} A_k x^{k*} + b^{k^T} x^{k*} = \min_{x \in \mathcal{R}^{S_k}} \left(x^{k^T} A_k x^k + b^{k^T} x^k \right), \; k = 1, 2, \ldots, K$$

$$x^* = \arg\min_{x \in \mathcal{R}^{S}} \left(x^T A x + b^T x \right) = \begin{bmatrix} x^{I*} \\ x^{II*} \\ \vdots \\ x^{K*} \end{bmatrix} = \begin{bmatrix} \arg\min_{x \in \mathcal{R}^{S_I}} \left(x^{I^T} A_I x^I + b^{I^T} x^I \right) \\ \arg\min_{x \in \mathcal{R}^{S_{II}}} \left(x^{II^T} A_{II} x^{II} + b^{II^T} x_{II} \right) \\ \vdots \\ \arg\min_{x \in \mathcal{R}^{S_K}} \left(x^{K^T} A_K x^K + b^{K^T} x^K \right) \end{bmatrix} = \begin{bmatrix} A_I^{-1} b^I \\ A_{II}^{-1} b^I \\ \vdots \\ A_K^{-1} b^K \end{bmatrix}$$

Separable constrains

$$x^* \to F(x^*) = \min_{x \in \mathscr{D}_x} F(x) \qquad F(x) = H\left(F_I(x^I), F_{II}(x^{II}), \ldots, F_K(x^K)\right)$$

$$x = \begin{bmatrix} x^I \\ x^{II} \\ \vdots \\ x^K \end{bmatrix}, \quad \begin{array}{l} x \in \mathscr{D}_x \subset \mathscr{R}^S, \ \mathscr{D}_x = \mathscr{D}_{x^I} \times \mathscr{D}_{x^{II}} \times \cdots \times \mathscr{D}_{x^K} \\ x^k \in \mathscr{D}_{x^k} \subset \mathscr{R}^{S_k}, \ k = I, II, \ldots, K, \ \sum_{k=I}^{K} S_k = S \\ \mathscr{D}_{x^k} = \left\{x^k \in \mathscr{R}^{S_k}; \varphi_l^k\left(x^k\right) = 0, \ l = 1, 2, \ldots, L_k, \ \psi_m^k\left(x^k\right) \le 0, \ m = 1, 2, \ldots, M_k\right\} \end{array}$$

$$\min_{x \in \mathscr{D}_x} F(x) = \min_{x \in \mathscr{D}_x} H\left(F_I(x^I), F_{II}(x^{II}), \ldots, F_K(x^K)\right) =$$

$$H\left(\min_{x^I \in \mathscr{D}_{x^I}} F_I(x^I), \min_{x_{II} \in \mathscr{D}_{x^{II}}} F_{II}(x^{II}), \ldots, \min_{x_K \in \mathscr{D}_{x^K}} F_K(x^K) \right)$$

Separable constrains

$$\min_{x \in \mathcal{D}_x} F(x) = \min_{x \in \mathcal{D}_x} H\left(F_I(x^I), F_{II}(x^{II}), \ldots, F_K(x^K)\right)$$

$$x^I \in \mathcal{D}_{x^I} \qquad x^{II} \in \mathcal{D}_{x^{II}} \qquad \cdots \qquad x^K \in \mathcal{D}_{x^k}$$

$$\min_{x^I \in \mathcal{D}_{x^I}} F_I(x^I) \qquad \min_{x_{II} \in \mathcal{D}_{x^{II}}} F_{II}(x^{II}) \qquad \cdots \qquad \min_{x_K \in \mathcal{D}_{x^K}} F_K(x^K)$$

$$x^{I*} \qquad x^{II*} \qquad x^{K*}$$

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Separable goal function and separable constrains with coordinate variable

$$x^* \rightarrow F(x^*) = \min_{x \in \mathscr{D}_x} F(x) \quad F(x) = H\left(F_I(x^I, w), F_{II}(x^{II}, w), \ldots, F_K(x^K, w)\right)$$

$$x \in D_x \subset R^S,\ D_x = \left\{x \in R^S;\ \varphi(x) = 0_L,\ \psi(x) \le 0_M\right\}$$

$$D_x = \left\{ x = \begin{bmatrix} x^I \\ x^{II} \\ \vdots \\ x^K \\ w \end{bmatrix} \in R^S, \varphi(x) = \begin{bmatrix} \varphi^I(x^I, w) \\ \varphi^{II}(x^{II}, w) \\ \vdots \\ \varphi^K(x^K, w) \\ \varphi^w(w) \end{bmatrix} = \begin{bmatrix} 0_{L_I} \\ 0_{L_{II}} \\ \vdots \\ 0_{L_K} \\ 0_{L_w} \end{bmatrix} = 0_L,\ \psi(x) = \begin{bmatrix} \psi^I(x^I, w) \\ \psi^{II}(x^{II}, w) \\ \vdots \\ \psi^K(x^K, w) \\ \psi^w(w) \end{bmatrix} \le \begin{bmatrix} 0_{M_I} \\ 0_{M_{II}} \\ \vdots \\ 0_{M_K} \\ 0_{M_w} \end{bmatrix} = 0_M \right\}$$

$$\sum_{k=I}^{K} L_k + L_w = L,\ \sum_{k=I}^{K} M_k + M_w = M$$

Separable goal function and separable constrains with coordinate variable

$$x=\begin{bmatrix} x^I \\ x^{II} \\ \vdots \\ x^K \\ w \end{bmatrix},\quad \begin{array}{c} x\in D_x\subset R^S,\, D_x=\bigcup\limits_{w\in D_w} D_{x^I}(w)\times D_{x^{II}}(w)\times\cdots\times D_{x^K}(w) \\ x^k\in D_{x^k}(w)\subset R^{S_k},\, w\in D_w\subset R^{S_w}\ k=I,II,\ldots,K,\ \sum\limits_{k=I}^{K} S_k+S_w=S \\ D_{x^k}(w)=\left\{x^k\in R^{S_k};\varphi_l^k(x^k,w)=0,\, l=1,2,\ldots,L_k,\,\psi_m^k(x^k,w)\le 0,\, m=1,2,\ldots,M_k\right\} \\ D_w=\left\{w\in R^{S_w};\varphi_l^w(w)=0,\, l=1,2,\ldots,L_w,\,\psi_m^w(w)=0,\, l=1,2,\ldots,M_w\right\} \end{array}$$

$$x^*\to F(x^*)=\min_{x\in\mathscr{D}_x} F(x)=\min_{x\in\mathscr{D}_x} H\left(F_I(x^I,w),F_{II}(x^{II},w),\ldots,F_K(x^K,w)\right)=$$

$$\min_{w\in\mathscr{D}_w} H\left(\min_{x^I\in\mathscr{D}_{x^I}(w)} F_I(x^I,w),\ \min_{x^{II}\in\mathscr{D}_{x^{II}}(w)} F_{II}(x^{II},w),\ldots,\ \min_{x^K\in\mathscr{D}_{x^K}(w)} F_K(x^K,w)\right)$$

Separable goal function and separable constrains with coordinate variable

The optimisation problem is solved in to two steps

Step I: For establish coordinate w we solve K optimisation tasks

$$x^{k*}(w) \rightarrow F_k\left(x^{k*}(w), w\right) = \min_{x^k \in D_{x^k}(w)} F_k\left(x^k, w\right),\ k = 1, 2, \ldots, K$$

Later on the optimal value of function is determined:

$$\overline{F}(w) \stackrel{\Delta}{=} H\left(F_I\left(x^{I*}(w), w\right), F_{II}\left(x^{II*}(w), w\right), \ldots, F_K\left(x^{K*}(w), w\right)\right)$$

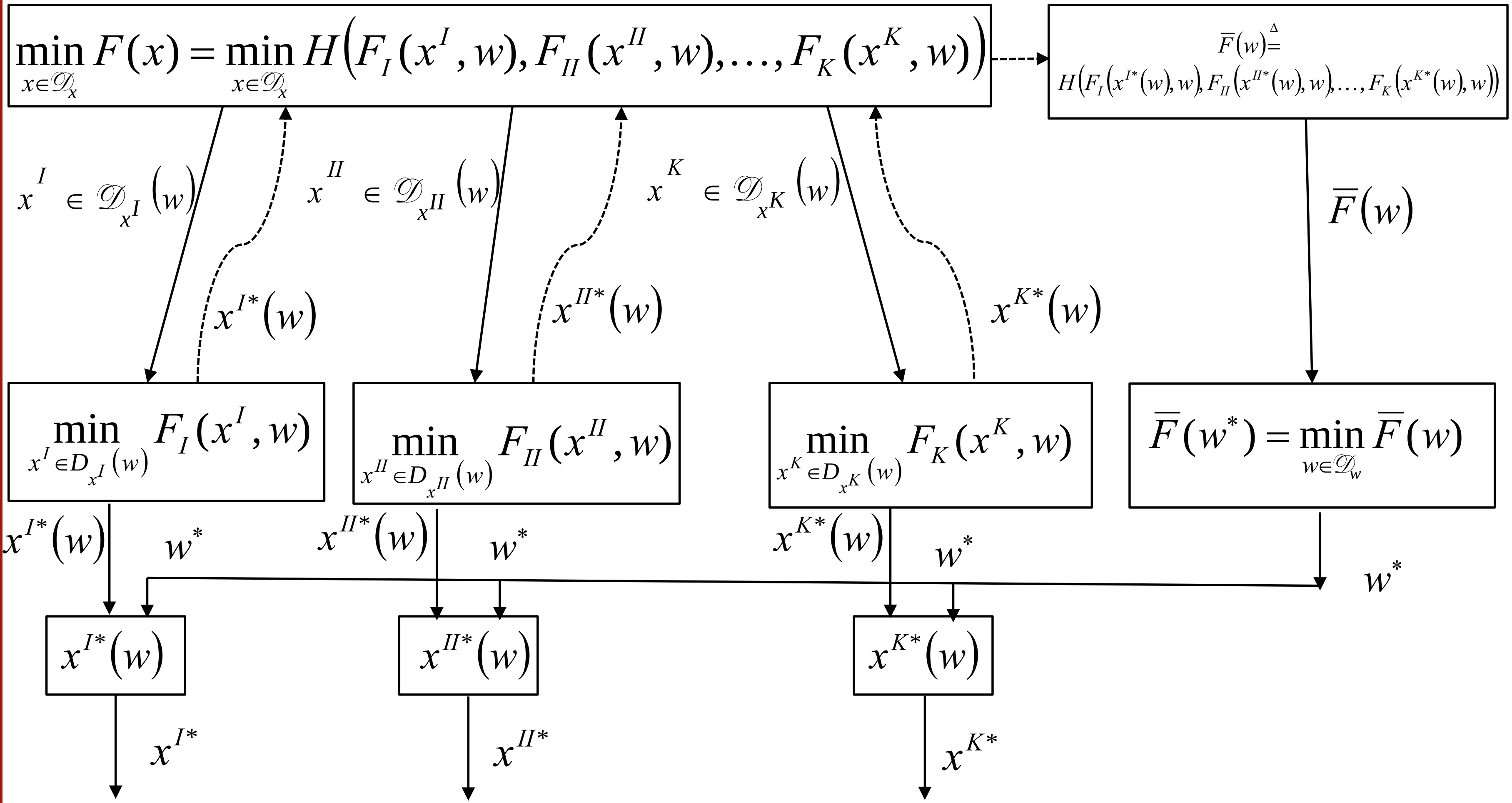
Step II: The optimal value of the coordinate variable is determined by solving :

$$w^* \rightarrow \overline{F}(w^*) = \min_{w \in \mathcal{D}_w} \overline{F}(w)$$

finally we obtain:

$$x^{k*} = x^{k*}\left(w^*\right),\ k = 1, 2, \ldots, K$$

Separable goal function and separable constrains with coordinate variable



Separable goal function and separable constrains with coordinate variable

$$x^* \rightarrow F(x^*) = \min_{x \in \mathscr{D}_x} F(x)$$

$$x \in \mathscr{D}_x \subset \mathscr{R}^S, \mathscr{D}_x = \{x \in \mathscr{R}^S; \varphi(x) = 0_L, \psi(x) \leq 0_M\}$$

$$F(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

$$D_x = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in R^4; \varphi(x) = \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \varphi_3(x) \end{bmatrix} = \begin{bmatrix} x_1 + x_3 - 3 \\ x_2 + x_4 - 1 \\ x_3 + x_4 - 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Przyjmijmy oznaczenia:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x^I \\ x^{II} \\ w_1 \\ w_2 \end{bmatrix}$$

Separable goal function and separable constrains with coordinate variable

Wówczas:

$$F(x) = \left(x^{I}\right)^2 + \left(x^{II}\right)^2 + \left(w_1\right)^2 + \left(w_2\right)^2;$$

$$F_I(x^{I}, w) = \left(x^{I}\right)^2 + \left(w_1\right)^2,$$

$$F_{II}(x^{II}, w) = \left(x^{II}\right)^2 + \left(w_2\right)^2$$

$$\mathscr{D}_{x^I}\left(w\right) = \left\{x^{I} \in \mathscr{R}^1; \varphi_1^{I}\left(x^{I}, w\right) = x^{I} + w_1 - 3 = 0\right\}$$

$$\mathscr{D}_{x^{II}}\left(w\right) = \left\{x^{II} \in \mathscr{R}^1; \varphi_1^{II}\left(x^{II}, w\right) = x^{II} + w_2 - 1 = 0\right\}$$

$$\mathscr{D}_{w} = \left\{ w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathscr{R}^2; \varphi_1^{w}\left(w\right) = w_1 + w_3 - 3 = 0 \right\}$$

Separable goal function and separable constrains with coordinate variable

Step I:

$$x^{I*}(w) \rightarrow F_I\left(x^{I*}(w), w\right) = \min_{x^I \in \mathscr{D}_{x^I}(w)} F_k\left(x^I, w\right) = \min_{x^I \in \mathscr{D}_{x^I}(w)} \left(\left(x^I\right)^2 + \left(w_1\right)^2\right)$$

$$x^{I*}(w) = 3 - w_1$$

$$x^{II*}(w) \rightarrow F_{II}\left(x^{II*}(w), w\right) = \min_{x^{II} \in \mathscr{D}_{x^{II}}(w)} F_k\left(x^{II}, w\right) = \min_{x^{II} \in \mathscr{D}_{x^{II}}(w)} \left(\left(x^{II}\right)^2 + \left(w_2\right)^2\right)$$

$$x^{I*}(w) = 1 - w_2$$

$$\overline{F}(w) \overset{\Delta}{=} H\left(F_I\left(x^{I*}(w), w\right), F_{II}\left(x^{II*}(w), w\right)\right) = \left(3 - w_1\right)^2 + \left(w_1\right)^2 + \left(1 - w_2\right)^2 + \left(w_2\right)^2$$

Step II:

$$w^* \rightarrow \overline{F}(w^*) = \min_{w \in \mathscr{D}_w} \overline{F}(w) = \min_{w \in \mathscr{D}_w} \left(\left(3 - w_1\right)^2 + \left(w_1\right)^2 + \left(1 - w_2\right)^2 + \left(w_2\right)^2\right)$$

$$w_1^* = 2,\ w_2^* = 1,$$

$$x^{I*}\left(w^*\right) = 3 - w_1^* = 1, \quad x^{II*}\left(w^*\right) = 1 - w_2^* = 0.$$

Thank you for attention