

Computer Science

Jerzy Świątek

Systems Modelling and Analysis

Choose yourself and new technologies

L.10. Multistage decision making



HUMAN CAPITAL
HUMAN – BEST INVESTMENT!



Wrocław University of Technology

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MULTISTAGE DECISION MAKING



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$$x_1^*, x_2^*, \dots, x_S^* \rightarrow F(x_1^*, x_2^*, \dots, x_S^*) = \min_{x_1, x_2, \dots, x_S \in \mathcal{D}_x} F(x_1, x_2, \dots, x_S)$$

$$\mathcal{D}_x = (x_1, x_2, \dots, x_S)$$

$$\left\{ \begin{aligned} [x_1 \ x_2 \ \dots \ x_S]^T \in \mathcal{R}^S : \varphi_l(x_1, x_2, \dots, x_S) = 0, l = 1, 2, \dots, L, \\ \psi_m(x_1, x_2, \dots, x_S) \leq 0, m = 1, 2, \dots, M \end{aligned} \right\}$$

The above task may be solved step by step, selecting a single decision variable to be optimized and relating with remaining decision variables.

Let us denote:

$$F \equiv F_S, \quad \varphi_l \equiv \varphi_{lS}, l = 1, 2, \dots, L, \quad \psi_m = \psi_{mS}, m = 1, 2, \dots, M \quad \mathcal{D}_x \equiv \mathcal{D}_{xS}$$



Multistage optimization

$$\text{Step 1. } x_S^* = G_S(x_1, \dots, x_{S-1}) \rightarrow F_S(x_1, x_2, \dots, x_S^*) = \min_{x_S \in \mathcal{D}_{x_S}} F_S(x_1, x_2, \dots, x_S)$$

The value of the goal function in the optimal solution:

$$F_{S-1}(x_1, x_2, \dots, x_{S-1}) \stackrel{\Delta}{=} F_S(x_1, x_2, \dots, x_S^*) = F_S(x_1, x_2, \dots, G_S(x_1, \dots, x_{S-1}))$$

Constraints in the optimal solution:

$$\mathcal{D}_{x_{S-1}}(x_1, \dots, x_{S-1}) \stackrel{\Delta}{=} \mathcal{D}_{x_S}(x_1, \dots, x_{S-1}, x_S^* = G_S(x_1, \dots, x_{S-1})) =$$

$$\left\{ \begin{array}{l} [x_1 \ x_2 \ \dots \ x_{S-1}]^T \in \mathcal{R}^{S-1} : \\ \varphi_{lS}(x_1, x_2, \dots, G_S(x_1, \dots, x_{S-1})) = \varphi_{lS-1}(x_1, x_2, \dots, x_{S-1}) = 0, \ l = 1, 2, \dots, L, \\ \psi_{mS}(x_1, x_2, \dots, G_S(x_1, \dots, x_{S-1})) = \psi_{mS-1}(x_1, x_2, \dots, x_{S-1}) \leq 0, \ m = 1, 2, \dots, M \end{array} \right\}$$



Multistage optimization

$$\text{Step 2. } x_{S-1}^* = G_{S-1}(x_1, \dots, x_{S-2}) \rightarrow F_{S-1}(x_1, x_2, \dots, x_{S-1}^*) = \min_{x_{S-1} \in \mathcal{D}_{x_{S-1}}} F_{S-1}(x_1, x_2, \dots, x_{S-1})$$

The value of the goal function in the optimal solution:

$$F_{S-2}(x_1, x_2, \dots, x_{S-2}) \stackrel{\Delta}{=} F_{S-1}(x_1, x_2, \dots, x_{S-1}^*) = F_{S-1}(x_1, x_2, \dots, G_{S-1}(x_1, \dots, x_{S-2}))$$

Constraints in the optimal solution:

$$\mathcal{D}_{x_{S-2}}(x_1, \dots, x_{S-2}) \stackrel{\Delta}{=} \mathcal{D}_{x_{S-1}}(x_1, \dots, x_{S-2}, x_{S-1}^* = G_{S-1}(x_1, \dots, x_{S-2})) =$$

$$\left\{ \begin{array}{l} [x_1 \ x_2 \ \dots \ x_{S-2}]^T \in \mathcal{R}^{S-2} : \\ \varphi_{lS-1}(x_1, x_2, \dots, G_{S-1}(x_1, \dots, x_{S-2})) = \varphi_{lS-2}(x_1, x_2, \dots, x_{S-2}) = 0, \ l = 1, 2, \dots, L, \\ \psi_{mS-1}(x_1, x_2, \dots, G_{S-1}(x_1, \dots, x_{S-2})) = \psi_{mS-2}(x_1, x_2, \dots, x_{S-2}) \leq 0, \ m = 1, 2, \dots, M \end{array} \right\}$$





⋮ Multistage optimization

Step S-1. $x_2^* = G_2(x_1) \rightarrow F_2(x_1, x_2^*) = \min_{x_2 \in \mathcal{D}_{x_2}} F_2(x_1, x_2)$

The value of the goal function in the optimal solution:

$$F_1(x_1) \stackrel{\Delta}{=} F_2(x_1, x_2^*) = F_2(x_1, G_2(x_1))$$

Constraints in the optimal solution:

$$\mathcal{D}_{x_1}(x_1) \stackrel{\Delta}{=} \mathcal{D}_{x_{S-1}}(x_1, x_2^* = G_1(x_1)) = \left\{ \begin{array}{l} x_1 \in \mathcal{R} : \\ \varphi_{l2}(x_1, G_1(x_1)) = \varphi_{l1}(x_1) = 0, l = 1, 2, \dots, L, \\ \psi_{m2}(x_1, G_{S-1}(x_1)) = \psi_{m1}(x_1) \leq 0, m = 1, 2, \dots, M \end{array} \right\}$$



Multistage optimization

$$\text{Step S-1. } x_1^* \rightarrow F_1(x_1^*) = \min_{x_1 \in \mathcal{D}_{x_1}} F_1(x_1)$$

We may now return to expressions „G” determined in the previous steps

$$x_1^*$$

$$x_2^* = G_2(x_1^*)$$

⋮

$$x_{S-1}^* = G_{S-1}(x_1^*, x_2^*, \dots, x_{S-1}^*)$$

$$x_S^* = G_S(x_1^*, x_2^*, \dots, x_{S-1}^*)$$



Multistage optimization

Example: $F(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2 = F_2(x_1, x_2)$

Step 1. $x_2^* = G_2(x_1) \rightarrow F_2(x_1, x_2^*) = \min_{x_2} \{ x_1^2 + x_1x_2 + x_2^2 \}$

$$\frac{\partial}{\partial x_2} F_2(x_1, x_2^*) = 2x_1 + x_2^* = 0 \Rightarrow x_2^* = -\frac{1}{2}x_1 = G_2(x_1)$$

$$F_1(x_1) \stackrel{\Delta}{=} F(x_1, x_2^*) = F(x_1, G_2(x_1)) = (x_1)^2 + x_1 \left(-\frac{1}{2}x_1 \right) + \left(-\frac{1}{2}x_1 \right)^2$$

$$F_1(x_1) = \frac{3}{4}x_1^2$$



Multistage optimization

Step 2. $x_1^* \Rightarrow F_1(x_1) = \min_{x_1} \left\{ \frac{3}{4} x_1^2 \right\}$

$$\frac{d}{dx_1} F_1(x_1) = 2 \frac{3}{4} x_1 = 0 \Rightarrow x_1^* = 0$$

Now we may return to initial condition:

$$x_1^* = 0$$

$$x_2^* = G_2(x_1^*) = -\frac{1}{2} x_1^* = 0$$



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DYNAMIC PROGRAMMING



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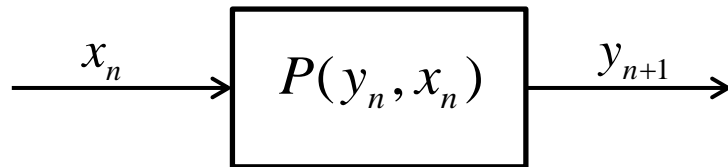


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Dynamic programming

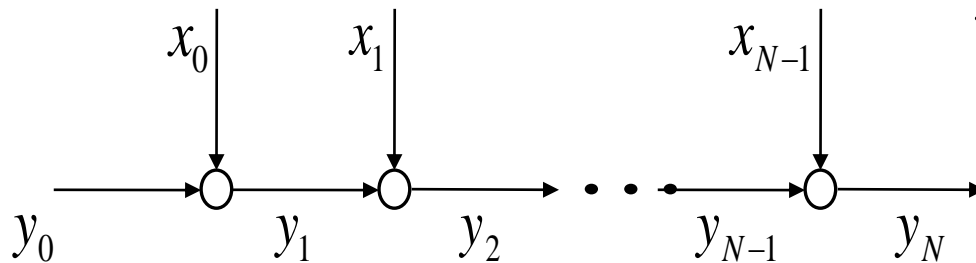
Dynamic process: $y_{n+1} = P(y_n, x_n), y_0$



n – the time step $n = 1, 2, \dots, N$

x_n – decision made in the n -th step

y_n – state of the process in the n -th step



Decision making task: to determine a sequence of decisions: $x_0^*, x_1^*, \dots, x_{N-1}^*$,

After N steps we want to achieve some goal, e.g.: $y_N = y^*$ – desired state,

The performance index $Q(x_0, \dots, x_{N-1}, y_1, \dots, y_N)$ must take minimal value



Dynamic programming

Let us evaluate the quality of the sequence of decisions and their effects

$$Q(x_0, x_1, \dots, x_{N-1}, y_1, y_2, \dots, y_N)$$

Examples.:

$$Q(x_0, x_1, \dots, x_{N-1}, y_1, y_2, \dots, y_N) = \sum_{n=0}^{N-1} x_n^2, \quad y_N = y^*$$

$$Q(x_0, x_1, \dots, x_{N-1}, y_1, y_2, \dots, y_N) = \sum_{n=0}^{N-1} x_n^2, \quad \underline{y} \leq y_N \leq \bar{y}$$

$$Q(x_0, x_1, \dots, x_{N-1}, y_1, y_2, \dots, y_N) = \sum_{n=1}^{N-1} (y_n - y_n^*), \quad 0 \leq x_n \leq \bar{x}$$

$$Q(x_0, x_1, \dots, x_{N-1}, y_1, y_2, \dots, y_N) = \sum_{n=0}^{N-1} A_{n+1}(x_n, y_{n+1}), \quad \text{with constraints } x_n \text{ and } y_n$$



Dynamic programming

The following procedure may be applied:

$$y_0$$

$$y_1 = P(y_0, x_0)$$

$$y_2 = P(y_1, x_1) = P(P(y_0, x_0), x_1) \stackrel{\Delta}{=} P_1(y_0, x_0, x_1)$$

$$\vdots$$

$$y_N = P(y_{N-1}, x_{N-1}) = P(P(y_{N-2}, x_{N-2}), x_{N-1}) = \dots \stackrel{y_{n+1}=P(y_n, x_n)}{=} P_{N-1}(y_0, x_0, x_1, \dots, x_N)$$

After substitution to $Q(\cdot)$ we obtain:

$$Q(x_0, x_1, \dots, x_{N-1}, y_1, y_2, \dots, y_N) = \sum_{n=0}^{N-1} A_{n+1}(x_n, y_{n+1}) \stackrel{\Delta}{=} F(y_0, x_0, x_1, \dots, x_{N-1})$$



Dynamic programming

Due to the fact, that:

$$Q(x_0, x_1, \dots, x_{N-1}, y_1, y_2, \dots, y_N) = \sum_{n=0}^{N-1} A_{n+1}(x_n, y_{n+1}) \stackrel{\Delta}{=} F(y_0, x_0, x_1, \dots, x_{N-1})$$

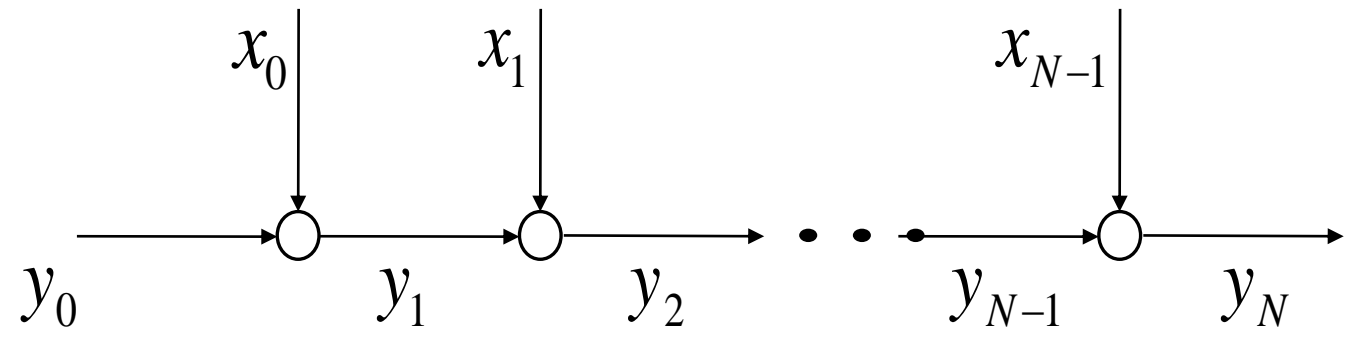
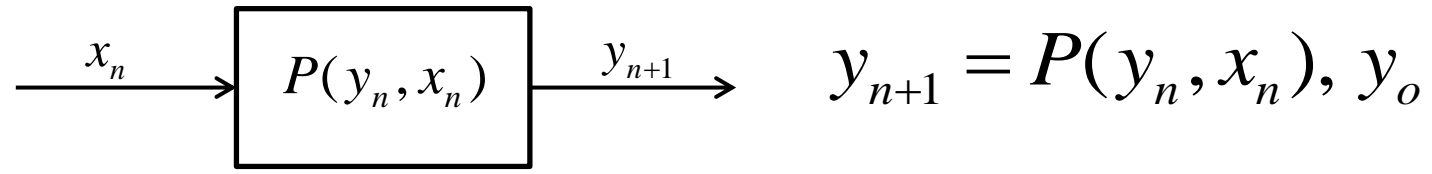
The optimization task:

$$x_0^*, x_1^*, \dots, x_{N-1}^* \rightarrow Q(x_0^*, x_1^*, \dots, x_{N-1}^*, y_1, y_2, \dots, y_N) = \min_{x_0, x_1, \dots, x_{N-1}} Q(x_0, x_1, \dots, x_{N-1}, y_1, y_2, \dots, y_N)$$

is equivalent to:

$$x_0^*, x_1^*, \dots, x_{N-1}^* \rightarrow F(y_0, x_0^*, x_1^*, \dots, x_{N-1}^*) = \min_{x_0, x_1, \dots, x_{N-1}} F(y_0, x_0, x_1, \dots, x_{N-1})$$

In order to solve the task above, the multi-stage approach may be applied.



$$Q(x_0, x_1, \dots, x_{N-1}, y_1, y_2, \dots, y_N) = \sum_{n=0}^{N-1} A_{n+1}(x_n, y_{n+1}) \stackrel{\Delta}{=} F(y_0, x_0, x_1, \dots, x_{N-1})$$



Dynamic programming

$$Q(x_0, x_1, \dots, x_{N-1}, y_1, y_2, \dots, y_N) = \sum_{n=0}^{N-1} A_{n+1}(x_n, y_{n+1}) \stackrel{\Delta}{=} F(y_0, x_0, x_1, \dots, x_{N-1})$$

Remark:

The final state y_N depends on decision x_{N-1} and the state y_{N-1} , which is dependent on previous decisions x_0, x_1, \dots, x_{N-2} .

The previous state y_{N-1} depends on decision x_{N-2} and the state y_{N-2} , which is dependent on previous decisions x_0, x_1, \dots, x_{N-3} .

⋮

The state y_2 depends on decision x_1 and the state y_1 , which is dependent on previous decisions x_0, x_1 .

The state y_1 depends on decision x_0 and the state y_0 , which values are known.

In order to solve the task above, the multi-stage approach may be applied. Taking into account the form of performance index (sum of functions of decisions x_n resulting states y_{n+1}).

Beginning from optimization of the last term we relate the solution from the previous state and previous decisions.



Dynamic programming

Step 1. $x_{N-1}^* \rightarrow \min_{x_{N-1}} A_N(x_{N-1}, y_N)$

We know, that: $y_N = P(y_{N-1}, x_{N-1})$

$$x_{N-1}^* = G_{N-1}(y_{N-1}) \rightarrow \min_{x_{N-1}} A_N(x_{N-1}, P(y_{N-1}, x_{N-1}))$$

$$\begin{aligned} V_{N-1}(y_{N-1}) &\stackrel{\Delta}{=} \min_{x_{N-1}} A_N(x_{N-1}, P(y_{N-1}, x_{N-1})) = \\ &= A_N(x_{N-1}^*, P(y_{N-1}, x_{N-1}^*)) = A_N(G_{N-1}(y_{N-1}), P(y_{N-1}, G_{N-1}(y_{N-1}))) \end{aligned}$$



Dynamic programming

$$\text{Step 2. } x_{N-2}^* \rightarrow \min_{x_{N-2}} \{A_{N-1}(x_{N-2}, y_{N-1}) + V_{N-1}(y_{N-1})\}$$

We know, that $y_{N-1} = P(y_{N-2}, x_{N-2})$

$$x_{N-2}^* = G_{N-2}(y_{N-2}) \rightarrow \min_{x_{N-2}} \{A_{N-1}(x_{N-2}, P(y_{N-2}, x_{N-2})) + V_{N-1}(P(y_{N-2}, x_{N-2}))\}$$

$$\begin{aligned} V_{N-2}(y_{N-2}) & \stackrel{\Delta}{=} \min_{x_{N-2}} \{A_{N-1}(x_{N-2}, P(y_{N-2}, x_{N-2})) + V_{N-1}(P(y_{N-2}, x_{N-2}))\} = \\ & = \{A_{N-1}(x_{N-2}^*, P(y_{N-2}, x_{N-2}^*)) + V_{N-1}(P(y_{N-2}, x_{N-2}^*))\} = \\ & = A_{N-1}(G_{N-2}(y_{N-2}), P(y_{N-2}, G_{N-2}(y_{N-2}))) + V_{N-1}(P(y_{N-2}, G_{N-2}(y_{N-2}))) \end{aligned}$$

⋮



Dynamic programming

$$\text{Step N-1. } x_1^* \rightarrow \min_{x_1} \{A_2(x_1, y_2) + V_2(y_2)\}$$

We know, that

$$y_2 = P(y_1, x_1)$$

$$x_1^* = G_1(y_1) \rightarrow \min_{x_1} \{A_2(x_1, P(y_1, x_1)) + V_2(P(y_1, x_1))\}$$

$$\begin{aligned} V_1(y_1) &\stackrel{\Delta}{=} \min_{x_1} \{A_2(x_1, P(y_1, x_1)) + V_2(P(y_1, x_1))\} = \\ &= A_2(x_1^*, P(y_1, x_1^*)) + V_2(P(y_1, x_1^*)) \\ &= A_2(G_1(y_1), P(y_1, G_1(y_1))) + V_2(P(y_1, G_1(y_1))) \end{aligned}$$



Dynamic programming

Step N.
$$x_0^* \rightarrow \min_{x_0} \{A_1(x_0, y_1) + V_1(y_1)\}$$

We know, that
$$y_1 = P(y_0, x_0)$$

$$x_0^* = G_0(y_0) \rightarrow \min_{x_0} \{A_1(x_0, P(y_0, x_0)) + V_1(P(y_0, x_0))\}$$

y_0 is known and from now on successive decisions may be determined

$$\begin{aligned}
 x_0^*, x_1^*, \dots, x_{N-1}^*, & \quad x_0^* = G_0(y_0) \rightarrow y_1 = P(y_0, x_0^*) \\
 & \quad x_1^* = G_1(y_1) \rightarrow y_2 = P(y_1, x_1^*) \\
 & \quad \vdots \\
 & \quad x_{N-2}^* = G_{N-2}(y_{N-2}) \rightarrow y_{N-1} = P(y_{N-2}, x_{N-2}^*) \\
 & \quad x_{N-1}^* = G_{N-1}(y_{N-1}) \rightarrow y_N = P(y_{N-1}, x_{N-1}^*)
 \end{aligned}$$



Dynamic programming

Example: $y_{n+1} = P(y_n, x_n) = 2y_n + x_n, \quad y_0 = 0$

$$Q(x_0, x_1, y_1, y_2) = \sum_{n=0}^1 A_{n+1}(x_n, y_{n+1}) = \sum_{n=0}^1 (x_n^2 + (y_{n+1} - 5)^2) =$$

$$= (x_0^2 + (y_1 - 5)^2) + (x_1^2 + (y_2 - 5)^2)$$

Step1.

$$x_1^* \rightarrow \min_{x_1} (x_1^2 + (y_2 - 5)^2)$$

$$y_2 = 2y_1 + x_1$$

$$x_1^* = G_1(y_1) \rightarrow \min_{x_1} (x_1^2 + (2y_1 + x_1 - 5)^2)$$

$$2x_1^* + 2(2y_1 + x_1^* - 5) = 0 \Rightarrow x_1^* = G_1(y_1) = \frac{5}{2} - y_1$$

$$V_1(y_1) = \left(\frac{5}{2} - y_1\right)^2 + \left(2y_1 + \frac{5}{2} - y_1 - 5\right)^2 = 2\left(y_1 - \frac{5}{2}\right)^2$$



Dynamic programming

Step 2.

$$x_0^* \rightarrow \min_{x_0} \left\{ (x_0^2 + (y_1 - 5)^2) + 2 \left(y_1 - \frac{5}{2} \right)^2 \right\}$$

$$y_1 = 2y_0 + x_0$$

$$x_0^* = G_0(y_0) \rightarrow \min_{x_0} \left\{ (x_0^2 + (2y_0 + x_0 - 5)^2) + 2 \left(2y_0 + x_0 - \frac{5}{2} \right)^2 \right\}$$

$$2x_0^* + 2(2y_0 + x_0^* - 5) + 4 \left(2y_0 + x_0^* - \frac{5}{2} \right) = 0 \Rightarrow x_0^* = G_0(y_0) = \frac{15}{8} - \frac{3}{2}y_0 = \frac{15}{8}$$

Now, we return back with calculations:

$$x_0^* = \frac{15}{8} \rightarrow y_1 = 2 \times 0 + \frac{15}{8} = \frac{15}{8}$$

$$x_1^* = \frac{5}{2} - \frac{15}{8} = \frac{5}{8} \rightarrow y_1 = 2 \times 0 + \frac{5}{8} = \frac{5}{8}$$



Thank you for attention

