Computer Science

Jerzy Świątek

Systems Modelling and Analysis

Choose yourself and new technologies

L.6 Constrained optimization methods

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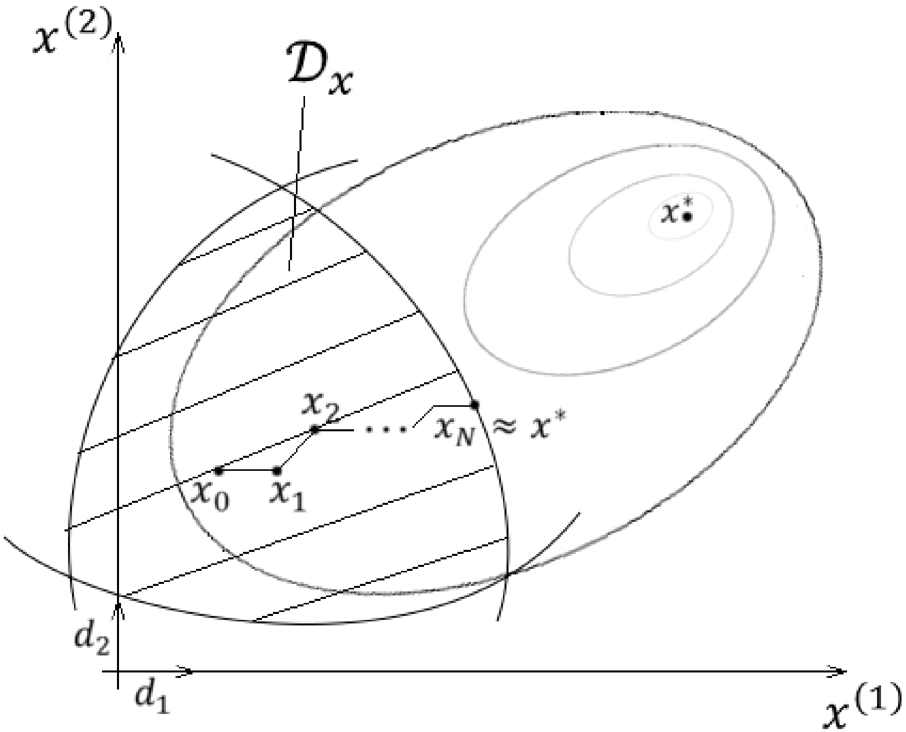
Numerical constrained optimization methods



$$x^* \to F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

1. Elimination of constraints
2. Penalty function method
   - exterior penalty
   - barrier function
3. Methods of feasible directions
4. Other approaches

Elimination of constraints

$$x^* \longrightarrow \min_{x \in D_x} F(x)$$

$$\uparrow \qquad\qquad\qquad \downarrow$$

$$p\colon R^s \to \mathcal{D}_x$$
$$x = p(z)$$
$$\bar{F}(z) = F\big(p(z)\big)$$
$$z \in R^s \to x \in \mathcal{D}_x$$

$$z^* \longleftarrow \min_{z \in R^s} \bar{F}(z)$$

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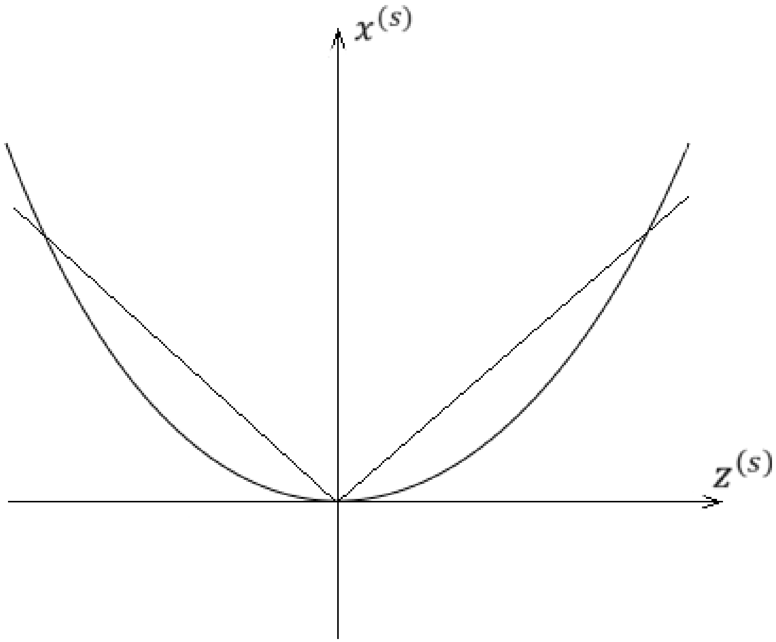
Example



$$D_x = \{x \in R^s, \quad x^{(s)} \geq 0, \quad s = 1, 2, \dots, S\}$$

$$x^{(s)} = \left(z^{(s)}\right)^2 \text{ or } x^{(s)} = \left|x^{(s)}\right|$$

$$z^{(s)} \in R^s \rightarrow x^{(s)} \in [0, \infty)$$

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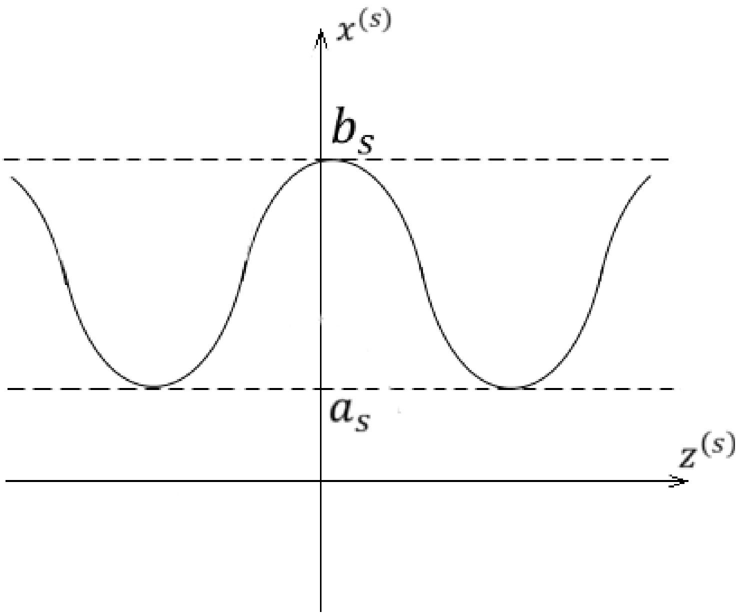
Example



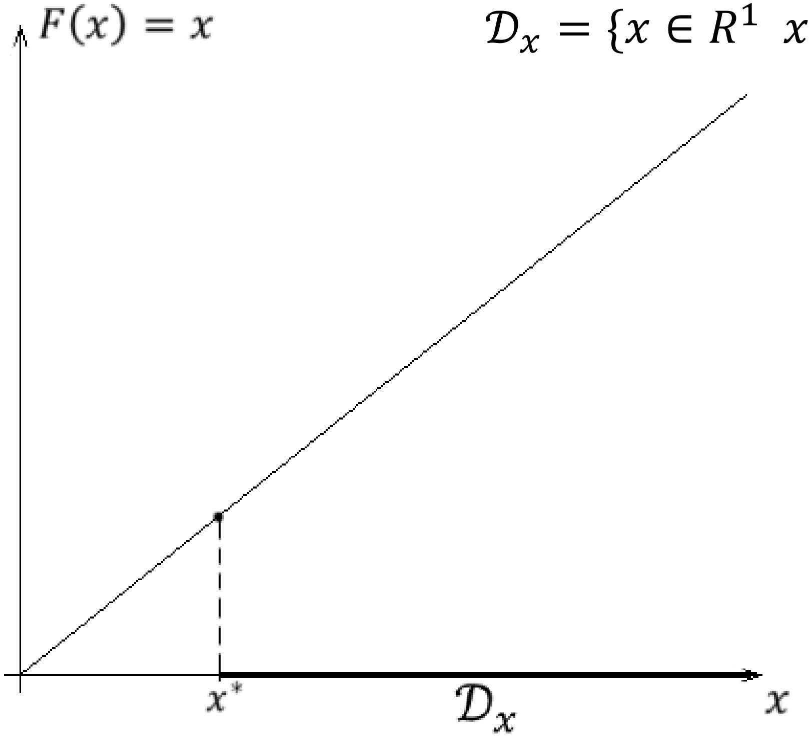
$$\mathcal{D}_x = \{x \in R^s, \quad a_s \le x^{(s)} \le b_s, \quad s = 1, 2, \dots, S\}$$

$$x^{(s)} = a_s + (b_s - a_s)\sin^2 z^{(s)}$$

$$z^{(s)} \in R^1 \to x^{(2)} \in [a_s, b_s]$$

Example

$$F(x) = x$$
$$\mathcal{D}_x = \{x \in R^1 \;\; x \geq 1\}$$



$$x^* \rightarrow \min_{x \geq 1} x$$

$$x = z^2 + 1 = p(z)$$

$$z \in R \rightarrow x \in [1, \infty)$$

$$\bar{F}(z) = z^2 + 1$$

$$z^* \rightarrow \min_{z \epsilon R} \bar{F}(z)$$

$$z^* \rightarrow \min_{z \epsilon R} (z^2 + 1)$$

$$(z^2 + 1)' = 2z = 0$$

$$z^* = 0$$

$$x^* = p(z^*) = (z^*)^2 + 1 = 1$$

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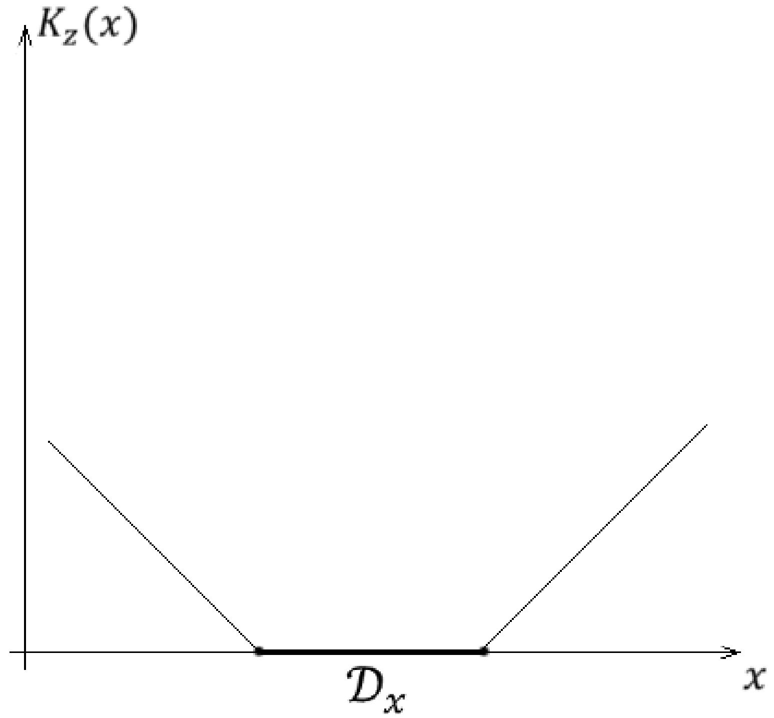
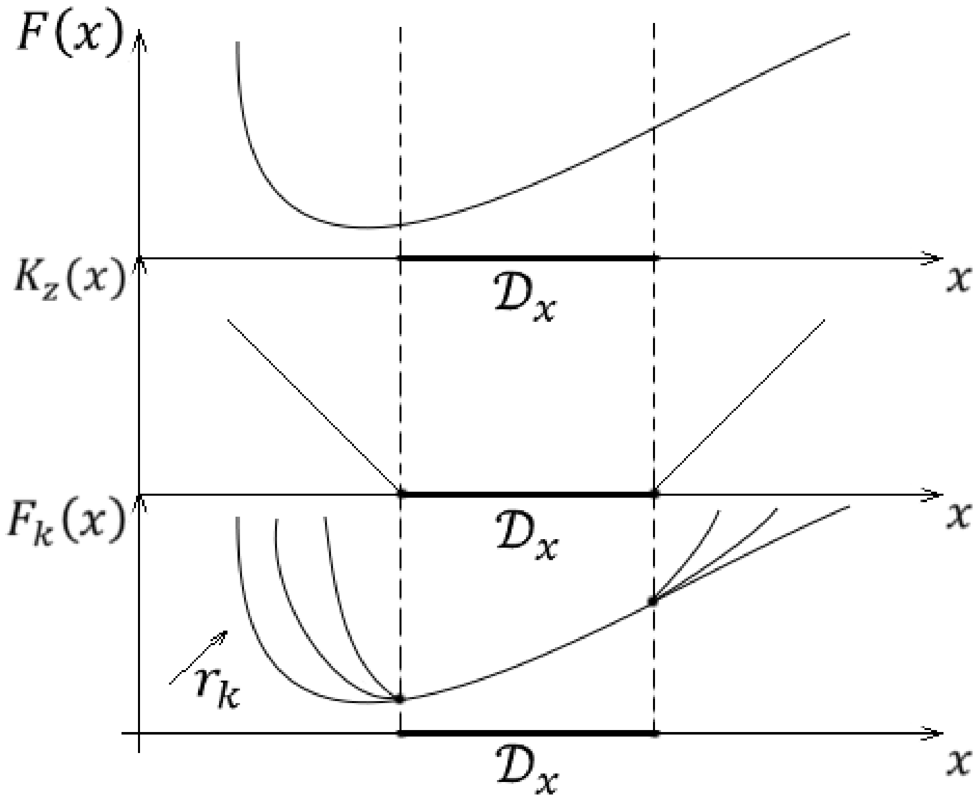
Penalty function method
Exterior penalty function

$$F_k(x) = F(x) + r_k K_z(x)$$

$$K_z(x) \begin{cases} = 0 & x \in \mathcal{D}_x \\ > 0 & x \notin \mathcal{D}_x \end{cases}$$





$r_k > 0$ $\lim_{k \to \infty} r_k = \infty$

Example of exterior penalty function

$$\mathcal{D}_x = \{x \in R^s,\ \varphi_l(x) = 0, \quad l = 1, 2, \dots, L, \quad \psi_m(x) \le 0, \quad m = 1, 2, \dots, M\}$$

$$\varphi_l(x) \to K_{lz}(x) = \left(\varphi_l(x)\right)^2, l = 1, 2, \dots, L$$

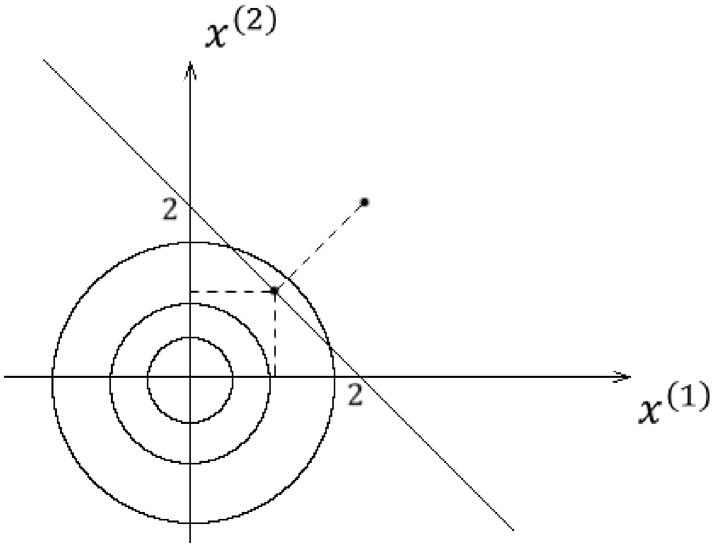
$$\psi_m(x) \to K_{mz}(x) = (\max\{0, \psi_m(x)\}) \quad , m = 1, 2, \dots, M$$

$$K_w(x) = \sum_{l=1}^{L} r_l\left(\varphi_l(x)\right)^2 + \sum_{m=1}^{M} \rho_m \ \max\{0, \varphi_m(x)\}$$

Example

$$F(x) = \left(x^{(1)}\right)^2 + \left(x^{(2)}\right)^2$$

$$\mathcal{D}_x = \{x \in R^2, x^{(1)} + x^{(2)} - 2 = 0\}$$



$$F_k(x) = \left(x^{(1)}\right)^2 + \left(x^{(2)}\right)^2 + r_k\left(x^{(1)} + x^{(2)} - 2\right)^2$$

$$F_k'(x) = 0$$

$$2x^{(1)} + 2r_k\left(x^{(1)} + x^{(2)} - 2\right) = 0$$

$$2x^{(2)} + 2r_k\left(x^{(1)} + x^{(2)} - 2\right) = 0$$

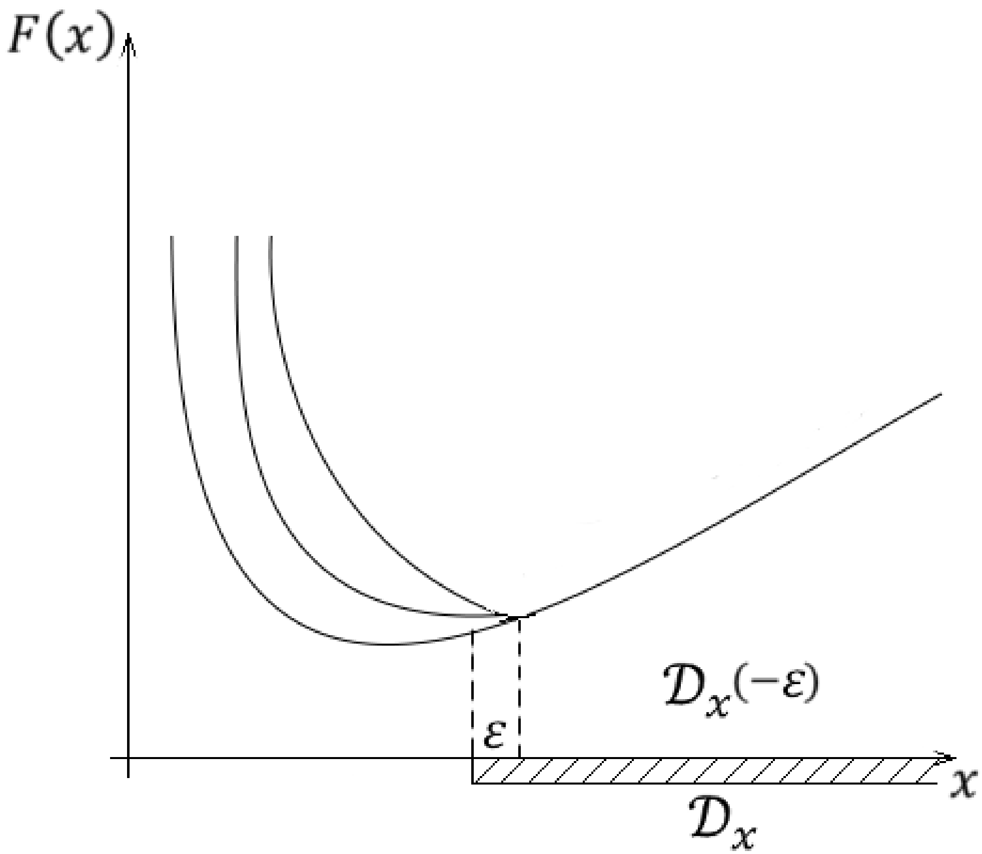
$$x^{(1)} = x^{(2)}$$

$$+2r_k\left(x^{(1)} + x^{(2)} - 2\right) = 0$$

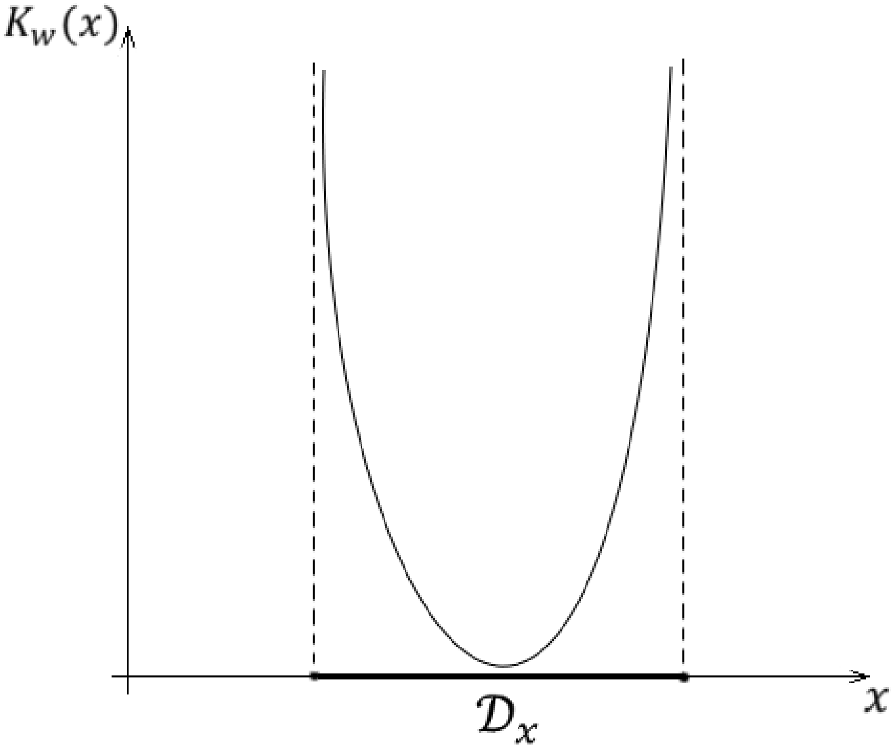
$$x^{(1)} = x^{(2)} = \frac{2r_k}{2r_k + 1}$$

$$x^{(1)} = x^{(2)} = \lim_{r_k \to \infty} \frac{2r_k}{2r_k + 1} = 1, \qquad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Modifications



Barrier function

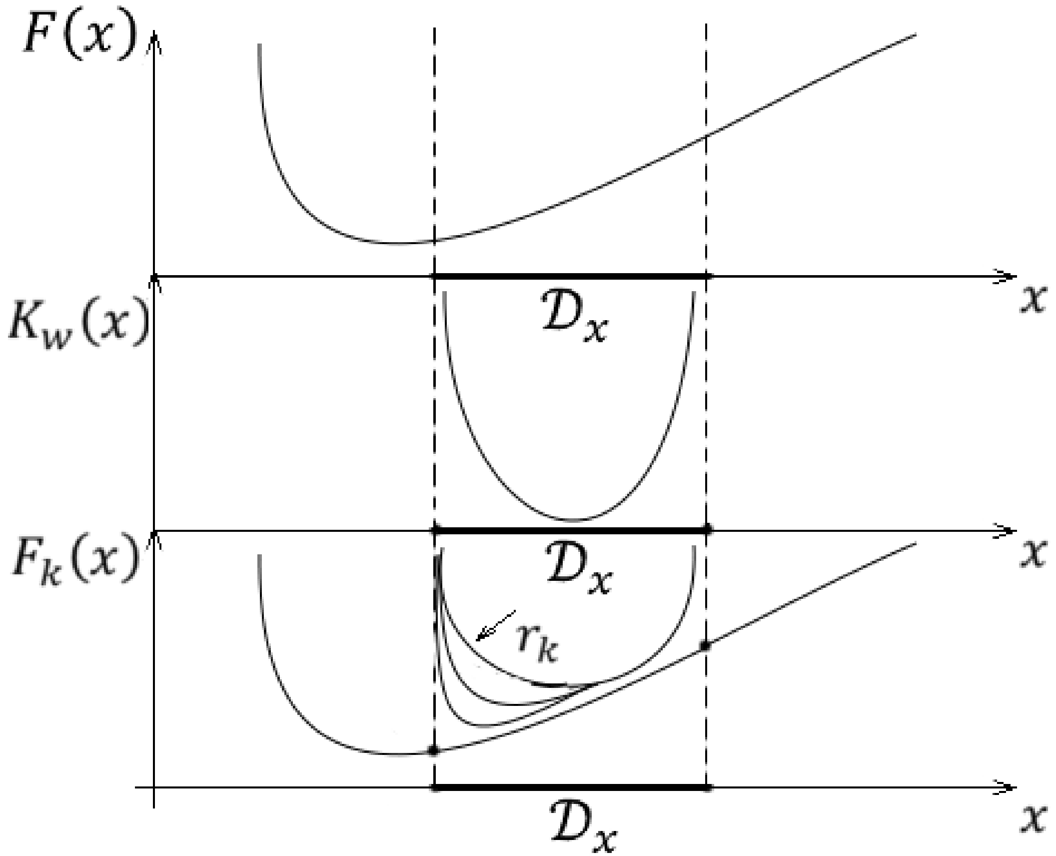


$$F_k(x) = F(x) + r_k K_w(x)$$

$K_w(x)$ – such a function, that

$\exists\ x_1, x_{,2}, \ldots, x_n \in \mathcal{D}_x \qquad \lim_{k\to\infty} x_k = x \in \mathcal{D}_x$

$\exists_k \quad K_w(x_k + 1) > K_w(x_k)$



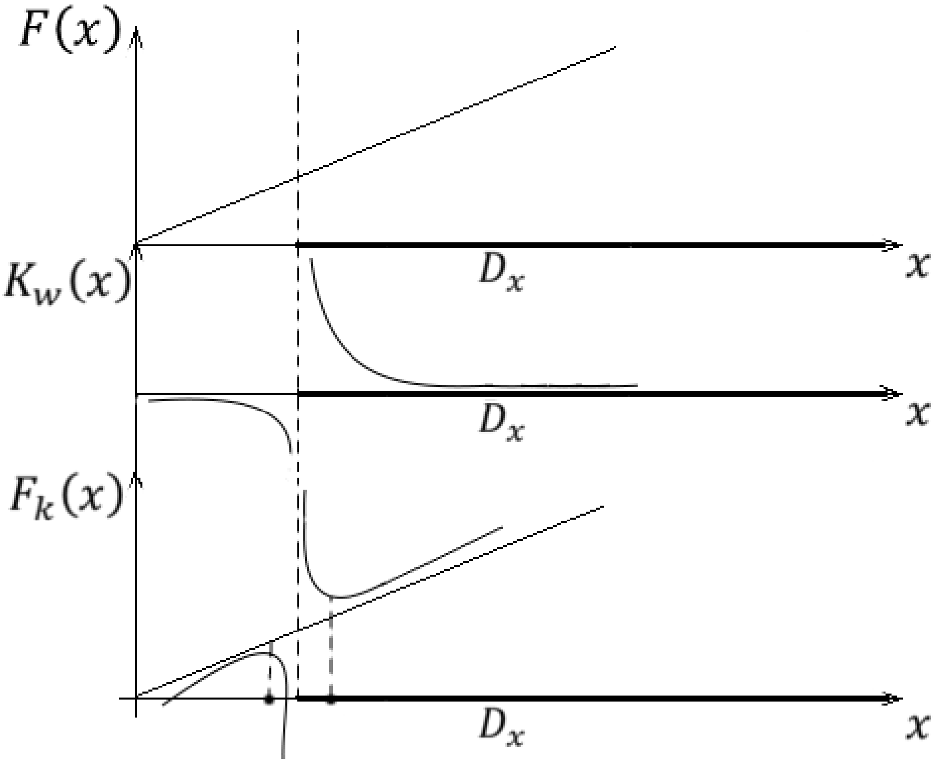
$$r_k > 0 \qquad \lim_{k \to \infty} r_k = 0$$

$$\mathcal{D}_x = \{x \in R^s, \qquad \psi_m(x) \le 0, \qquad m = 1, 2, \dots, M\}$$

$$K_w(x) = \sum_{m=1}^{M} \frac{-r_m}{\psi_m(x)}, \quad m = 1, 2, \dots, M$$

Example

$$F(x) = x;\ \mathcal{D}_x = \{x \in R^1, x \geq 1\} \equiv \{x \in R^1, 1 - x \leq 0\}$$



$$K_w(x) = \frac{-1}{1-x} = \frac{1}{x-1}$$

$$F_k(x) = x + r_k \frac{1}{x-1}$$

$$F'(x) = 1 + \frac{-r_k}{(x-1)^2} = 0$$

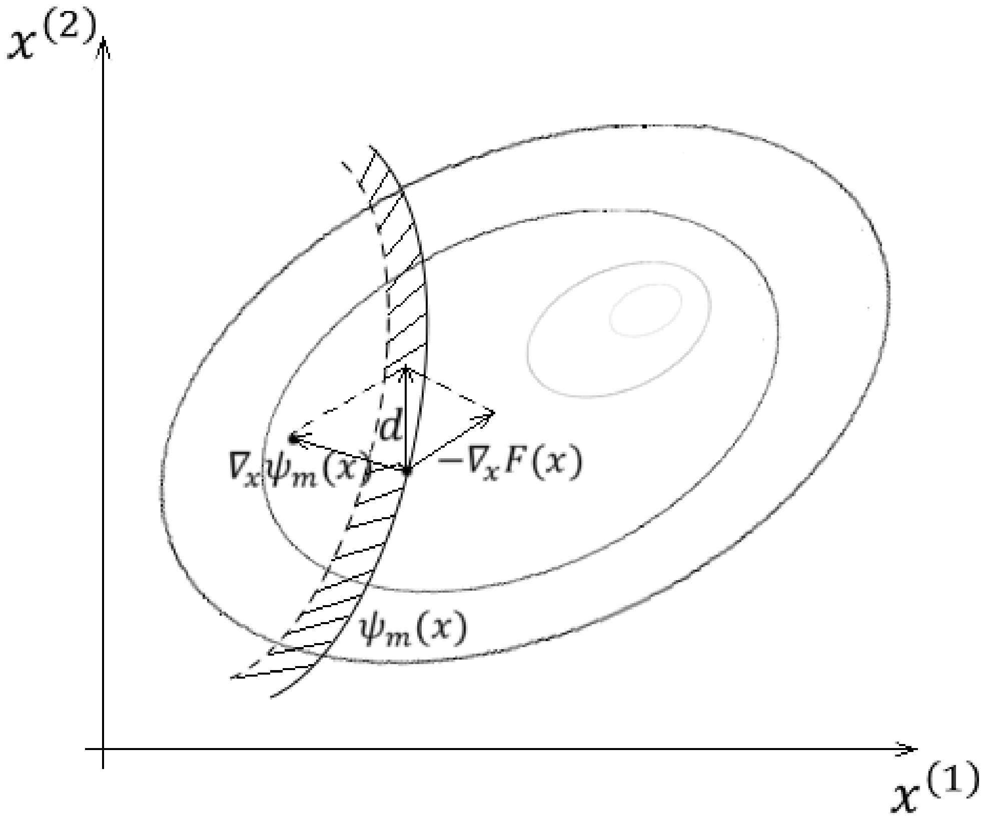
$$x_k = 1 \pm \sqrt{r_k}$$

$$x_k = 1 - \sqrt{r_k} \notin \mathcal{D}_x$$

$$x_k = 1 + \sqrt{r_k} \in \mathcal{D}_x$$

$$\lim_{n \to \infty} x_n = 1$$

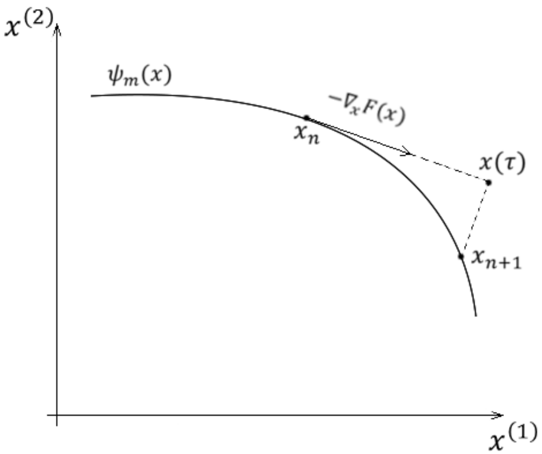
Feasible directions method

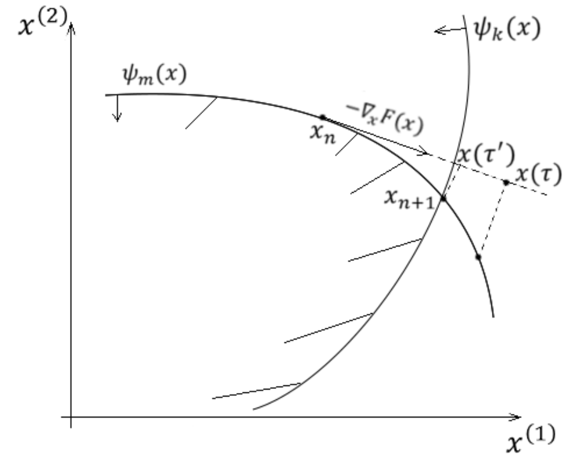


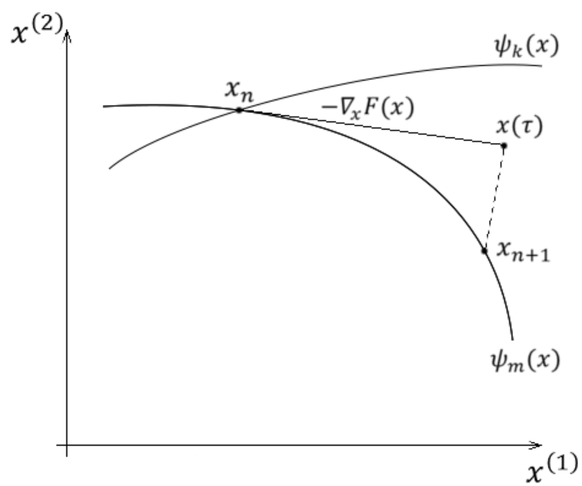
$$d = \frac{\nabla_x \psi_m(x)}{\|\nabla_x \psi_m(x)\|} - \frac{\nabla_x F(x)}{\|\nabla_x F(x)\|}$$

$$x: \psi(x) - \delta \leq 0$$

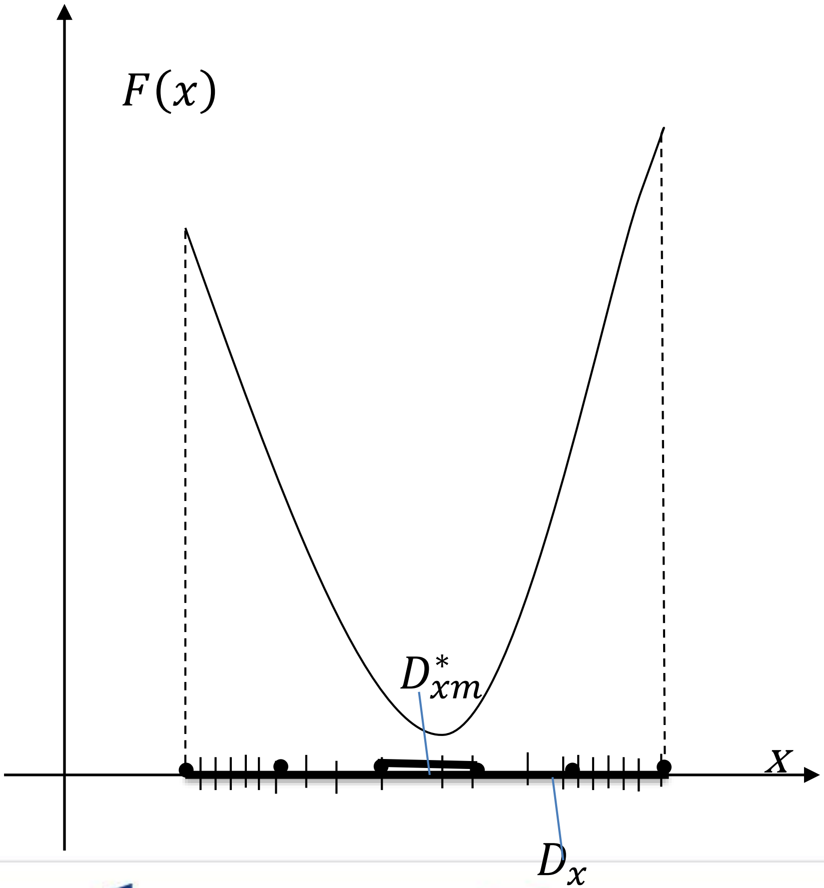
Gradient projection method of Rosen







Random search



$$f(x) = \frac{F(x)}{\int_{D_x} F(x)\,dx}$$

Random search

Data: $F(x), D_x, \varepsilon, N, M$

Step 1: Generate N point in set D_x with probability density

$$f(x) = \frac{F(x)}{\int_{D_x} F(x)dx}$$

Step 2: Divide set D_x for M disjoint sets such that

$$D_x = \dot{\bigcup_{m=1}^{M}} D_{xm}, \quad \|D_{xm}\| = \frac{1}{M}\|D_x\|$$

Step 3. Count points in all sets D_{xm}

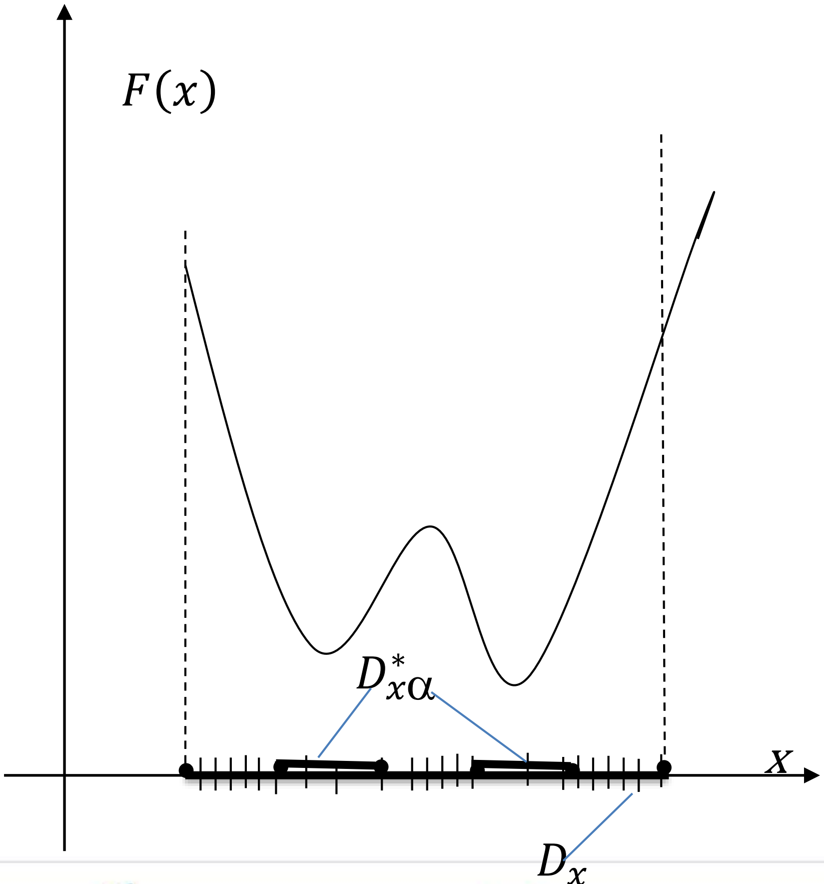
$$N_m - number\ of\ po\text{int}\, s\ in\ the\ set\ D_{xm}$$

Step 4. For the next division choose such set that

$$D_{xm}^* \to N_m^* = \min_{1 \le m \le M}\{N_m\}\ if\ \|D_{xm}\| < \varepsilon\ stop, \forall po\text{int} \in D_{xm}^*\ solution$$

Otrherwise i.e. $\|D_{xm}^*\| \ge \varepsilon\ in\ the\ place\ D_x := D_{xm}^*$ and go to step 1.

Random search



$$f(x) = \frac{F(x)}{\int_{D_x} F(x)dx}$$

Random search

Data: $F(x), D_x, \varepsilon, N, M$

Step 1: Generate N point in set D_x with probability density

$$f(x) = \frac{F(x)}{\int_{D_x} F(x)dx}$$

Step 2: Divide set D_x for M disjoint sets such that

$$D_x = \dot{\bigcup_{m=1}^{M}} D_{xm}, \quad \|D_{xm}\| = \frac{1}{M}\|D_x\|$$

Step 3. Count points in all sets D_{xm}

$$N_m - number\ of\ po\operatorname{int}s\ in\ the\ set\ D_{xm}$$

Step 4. For the next division choose such set that

$$D^*_{xm\alpha} \to N^*_{m\alpha} \le \alpha, D^*_{x\alpha} = \bigcup_m D^*_{xm\alpha}\ if\ \left\|D^*_{x\alpha}\right\| < \varepsilon\ stop, \forall po\operatorname{int} \in D^*_{x\alpha}\ solution$$

Otrherwise i.e. $\left\|D^*_{xm}\right\| \ge \varepsilon\ in\ the\ place\ D_x := D^*_{xm}$ and go to step 1.