

Computer Science

Jerzy Świątek

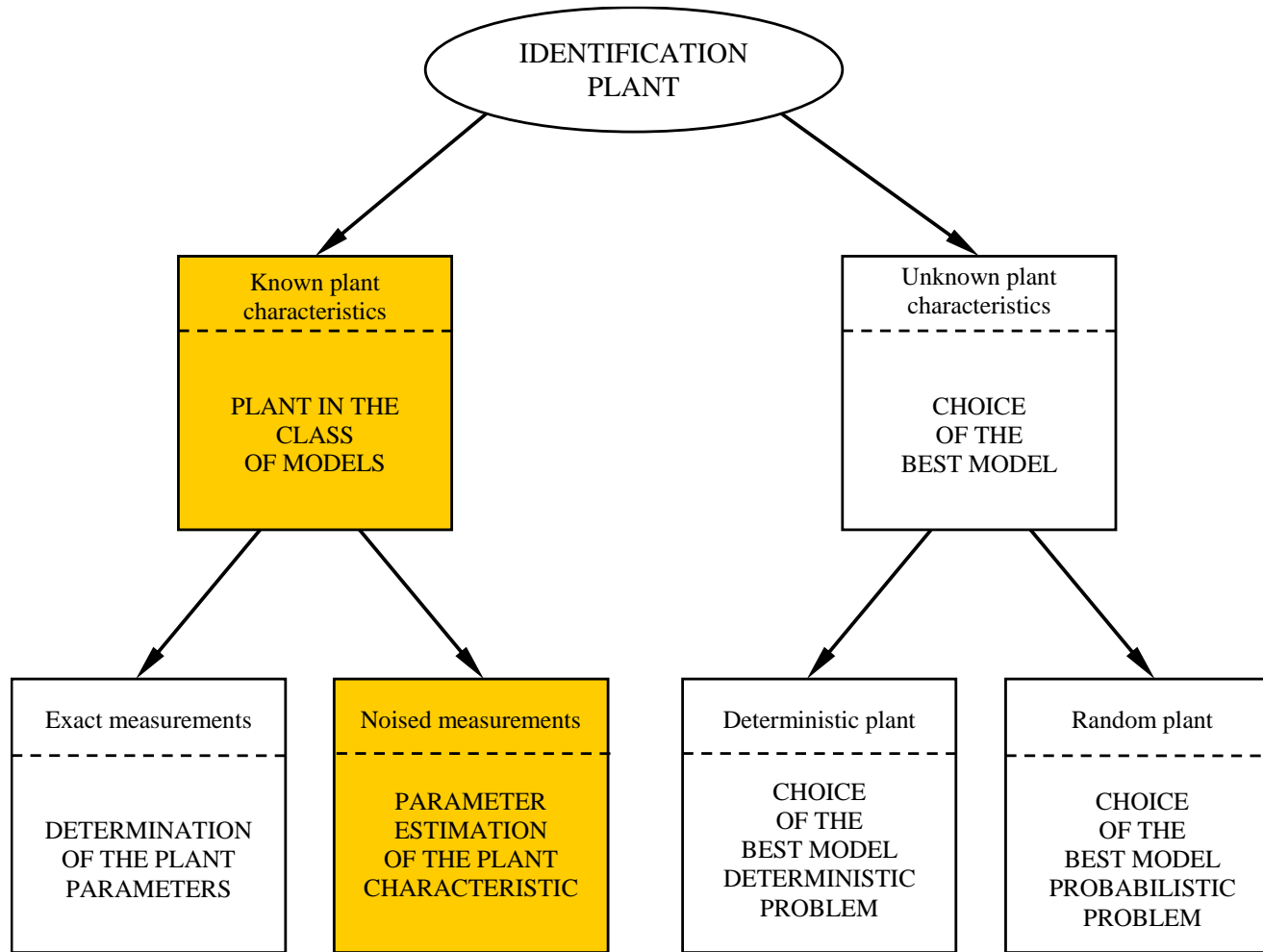
Systems Modelling and Analysis

Choose yourself and new technologies

L.3b. Estimation of the plant parameter with random value

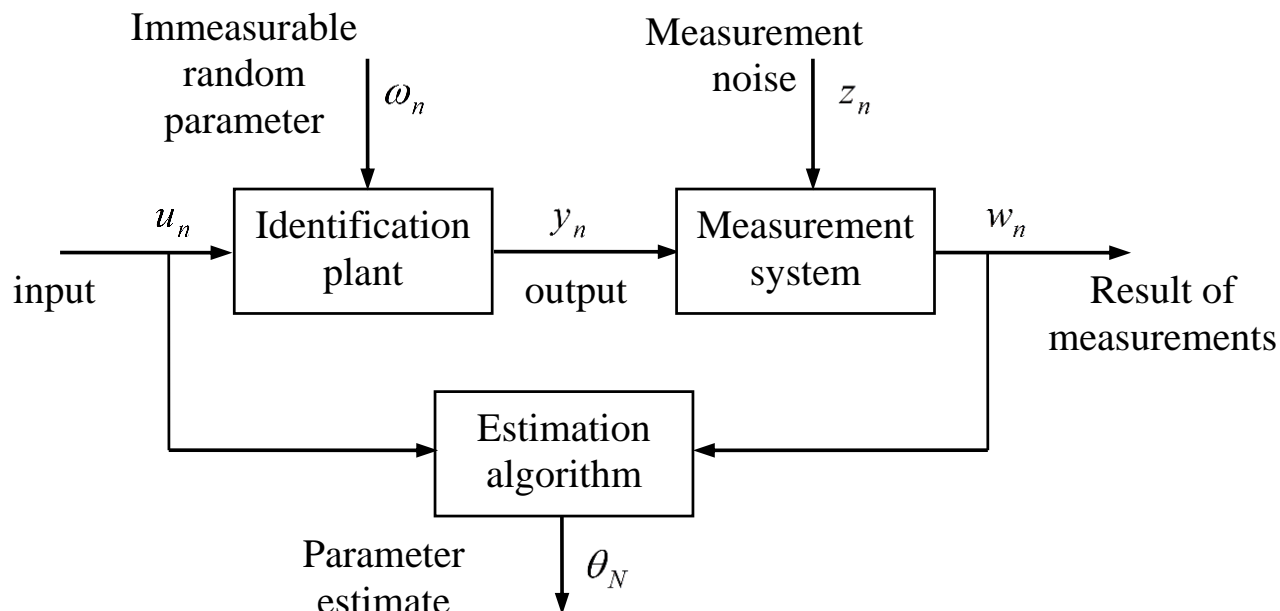


Project co-financed from the EU European Social Fund





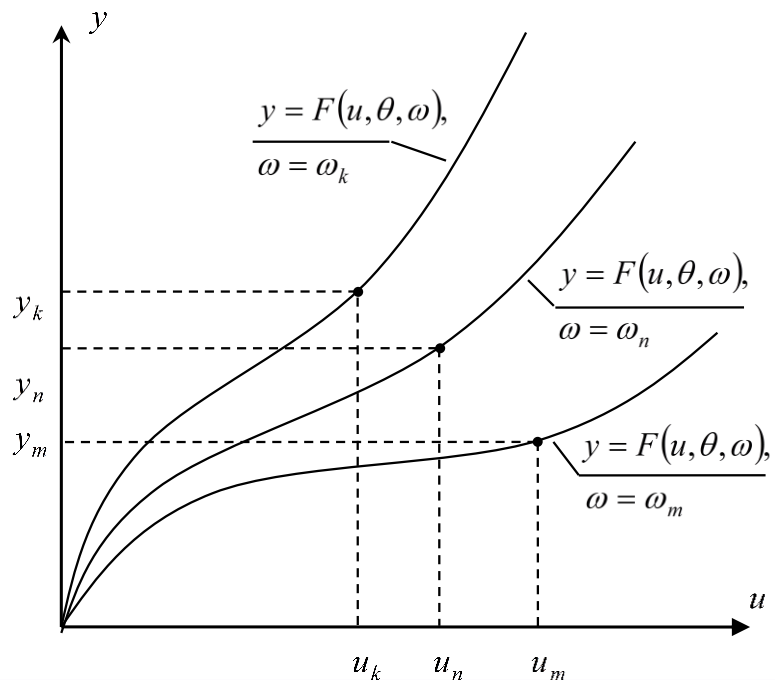
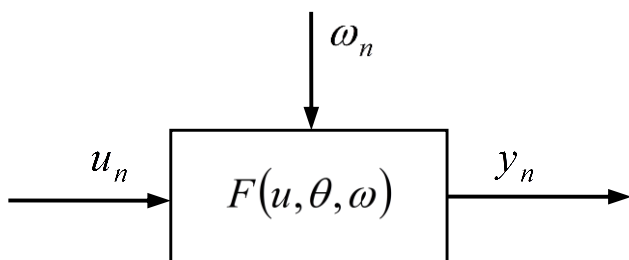
Plant parameter estimation problem





Immeasurable random plant parameter

- Measurements of plant characteristic with random parameter





Immeasurable random plant parameter

- Problem formulation

Plant characteristic: $y = F(u, \theta, \omega)$

Random plant parameter: $\omega \in \Omega \subseteq \mathcal{R}^L$, ($\dim y = \dim \omega = L$)

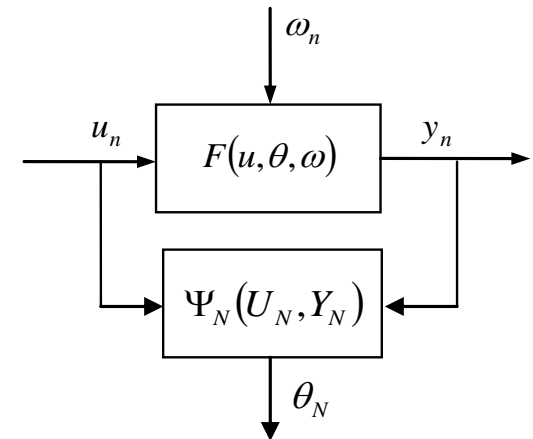
F – one-to-one mapping $\omega = F_{\omega}^{-1}(u, \theta, y)$

ω_n – value of random variable ω from $\Omega \subseteq \mathcal{R}^L$

Probability density function $f_{\omega}(\omega)$ is given

Measurements: $U_N = [u_1 \ u_2 \ \dots \ u_N]$, $Y_N = [y_1 \ y_2 \ \dots \ y_N]$

Estimation algorithm: $\theta_N = \Psi_N(U_N, Y_N)$



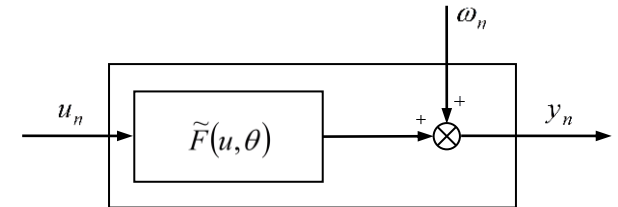


Least square method

Assumptions:

$$y = F(u, \theta, \omega) = \tilde{F}(u, \theta) + \omega \quad \text{– plant characteristic}$$

$$E[\underline{\omega}] = 0 \quad \text{Var}[\underline{\omega}] < \infty$$



Calculations:

Least square method minimizes empirical variance:

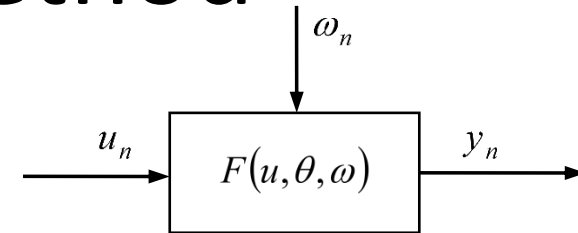
$$\text{Var}_{\omega N}(U_N, Y_N, \theta) = \frac{1}{N} \sum_{n=1}^N (y_n - \tilde{F}(u_n, \theta))^2$$

Estimation algorithm has the form:

$$\theta_N = \Psi_N(U_N, Y_N) \rightarrow \text{Var}_{\omega N}(U_N, Y_N, \theta_N) = \min_{\theta \in \Theta} \text{Var}_{\omega N}(U_N, Y_N, \theta)$$



Maximum likelihood method



Assumptions:

ω_n is value of random variable $\underline{\omega}$, with probability density function $f_{\omega}(\omega)$

For a given input u_n , output $y_n, n = 1, 2, \dots, N$ is measured

Sequence $y_n, n = 1, 2, \dots, N$ contains values of random variable \underline{y} : $\underline{y} = F(u, \theta, \underline{\omega})$

Calculations:

Probability density function $f_y(y, \theta; u) = f_{\omega}(F_{\omega}^{-1}(u, \theta, y)) \cdot |J_F|$

where J_F is Jacobi matrix: $J_F = \frac{\partial F_{\omega}^{-1}(u, \theta, y)}{\partial y}$

Likelihood function: $L_N(Y_N, \theta; U_N) = \prod_{n=1}^N f_y(y_n, \theta; u_n) = \prod_{n=1}^N f_{\omega}(F_{\omega}^{-1}(u_n, \theta, y_n)) |J_F|$

Estimation algorithm: $\theta_N = \Psi_N(U_N, Y_N) \rightarrow L_N(Y_N, \theta_N; U_N) = \max_{\theta \in \Theta} L_N(Y_N, \theta; U_N)$



Bayesian method

Assumptions:

Additionally θ is value of random variable $\underline{\theta}$, with probability density function $f_{\theta}(\theta)$

Calculations:

$$r(\bar{\theta}, Y_N; U_N) \stackrel{df}{=} E_{\underline{\theta}}[L(\underline{\theta}, \bar{\theta}) | Y_N; U_N] = \int_{\Theta} L(\theta, \bar{\theta} = \bar{\Psi}(U_N, Y_N)) f'(\theta | Y_N; U_N) d\theta$$

$$f_y(y | \theta; u) = f_{\omega}(F_{\omega}^{-1}(u, \theta, y)) |J_F|$$

A'posteriori probability density function:

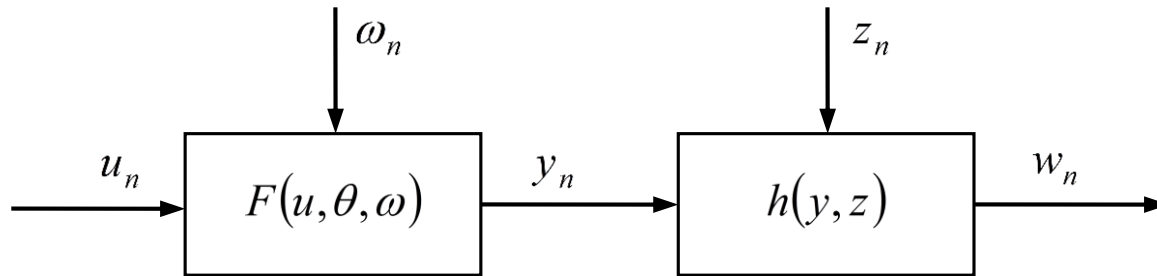
$$f'(\theta | Y_N; U_N) = \frac{f_{\theta}(\theta) \cdot \prod_{n=1}^N f_{\omega}(F_{\omega}^{-1}(u_n, \theta, y_n)) |J_F|}{\int_{\Theta} f_{\theta}(\theta) \cdot \prod_{n=1}^N f_{\omega}(F_{\omega}^{-1}(u_n, \theta, y_n)) |J_F| d\theta}$$

Estimation algorithm: $\theta_N = \Psi_N(U_N, Y_N) \rightarrow r(\theta_N, Y_N; U_N) = \min_{\bar{\theta} \in \Theta} r(\bar{\theta}, Y_N; U_N)$



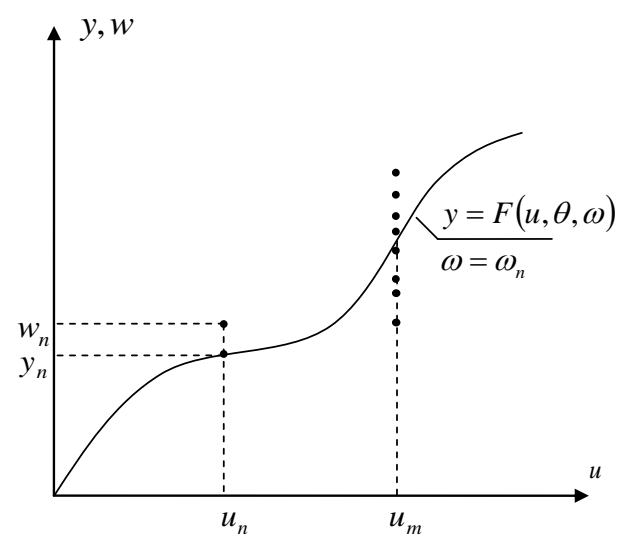
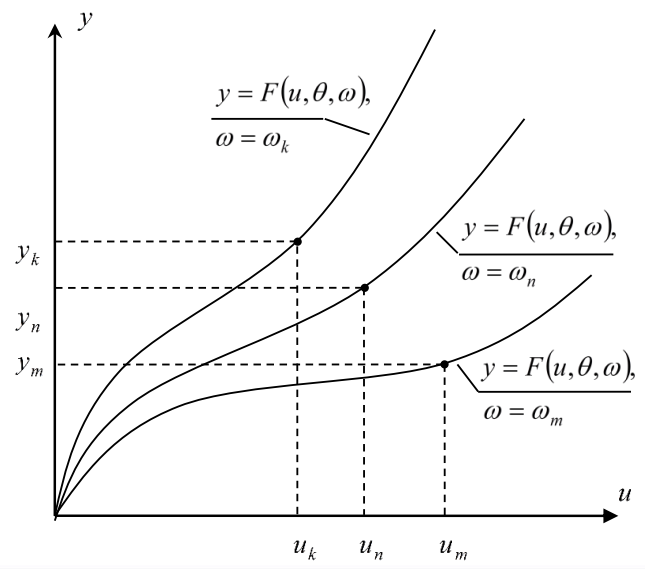
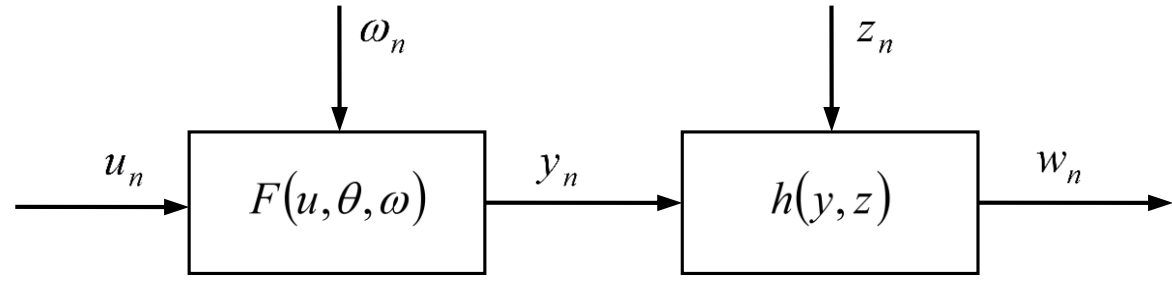
Random plant parameter and measurement noise

- Noised measurement plant output with randomly changed parameters



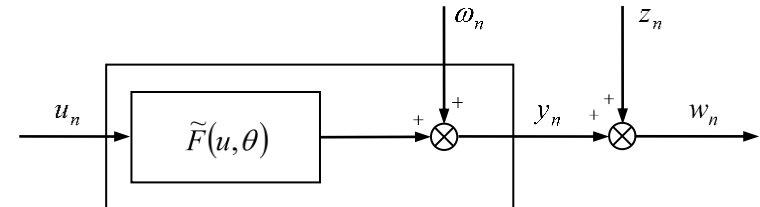


- Noised measurement plant output with randomly changed parameters





Least square method



Assumptions:

\underline{z} , $\underline{\omega}$ – random variables such that $E[\underline{\omega}] = 0$, $Var[\underline{\omega}] < \infty$, $E[\underline{z}] = 0$, $Var[\underline{z}] < \infty$

Measurements are values of random variable: $\underline{w} = \tilde{F}(u, \theta) + \underline{\omega} + \underline{z}$

Calculations:

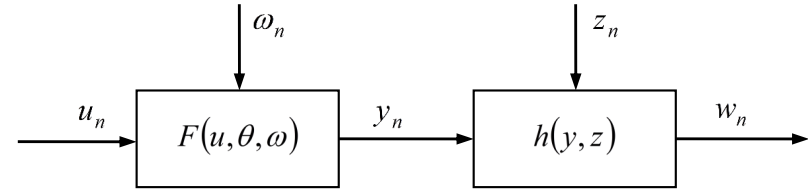
$$E[\underline{\omega} + \underline{z}] = 0, \quad Var[\underline{\omega} + \underline{z}] < \infty$$

Empirical variance: $Var_{(\omega+z)N}(U_N, W_N, \theta) = \frac{1}{N} \sum_{n=1}^N (w_n - \tilde{F}(u_n, \theta))^2$

Estimation algorithm: $\theta_N = \Psi_N(U_N, Y_N) \rightarrow Var_{(\omega+z)N}(U_N, W_N, \theta_N) = \min_{\theta \in \Theta} Var_{(\omega+z)N}(U_N, W_N, \theta)$



Maximum likelihood method



Assumptions:

ω_n is value of random variable $\underline{\omega}$, with probability density function $f_{\omega}(\omega)$

z_n is value of random variable \underline{z} , with probability density function $f_z(z)$

$w_n, n = 1, 2, \dots, N$ are values of random variable $\underline{w} = h(\underline{y}, \underline{z})$

For a given $u_n, y_n, n = 1, 2, \dots, N$ are values of random variable $\underline{y} = F(u, \theta, \underline{\omega})$

Calculations:

Function h is no one to one mapping with respect $(\underline{y}, \underline{z})$, so we add identity $\underline{y} = \underline{y}$ and we have:

$$\underline{w} = h(\underline{y}, \underline{z})$$

$$\underline{y} = \underline{y}$$

The inverse transformation:

$$\underline{z} = h_z^{-1}(\underline{y}, \underline{w})$$

$$\underline{y} = \underline{y}$$



Maximum likelihood method

Jacobi matrix:
$$J = \begin{bmatrix} \frac{\partial h_z^{-1}(y, w)}{\partial w} & \frac{\partial h_z^{-1}(y, w)}{\partial y} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial y} \\ \mathbf{O}_{L \times L} & \mathbf{I}_{L \times L} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_z^{-1}(y, w)}{\partial w} & \frac{\partial h_z^{-1}(y, w)}{\partial y} \\ \mathbf{O}_{L \times L} & \mathbf{I}_{L \times L} \end{bmatrix}$$

where: $\mathbf{O}_{L \times L}$ is L – dimensional zero matrix, $\mathbf{I}_{L \times L}$ is L – dimensional unit matrix

Determinant of Jacobi matrix:

$$|J| = \left| \begin{bmatrix} \frac{\partial h_z^{-1}(y, w)}{\partial w} & \frac{\partial h_z^{-1}(y, w)}{\partial y} \\ \mathbf{O}_{L \times L} & \mathbf{I}_{L \times L} \end{bmatrix} \right| = \left| \frac{\partial h_z^{-1}(y, w)}{\partial w} \right| = |J_h|$$



Maximum likelihood method

Joint probability density function of random variables (w, y) :

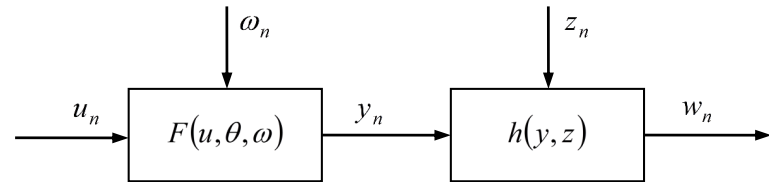
$$f_w(w, y, \theta; u) = f_z(h_z^{-1}(y, w))f_y(y, \theta; u)|J| = f_z(h_z^{-1}(y, w))f_\omega(F_\omega^{-1}(u, \theta, y))|J_F||J_h|$$

Marginal probability density function of random variable w :

$$f_w(w, \theta; u) = \int_{\mathcal{Y}} f_z(h_z^{-1}(y, w))f_y(y, \theta; u)|J|dy = \int_{\mathcal{Y}} f_z(h_z^{-1}(y, w))f_\omega(F_\omega^{-1}(u, \theta, y))|J_F||J_h|dy$$

Likelihood function:

$$L_N(W_N, \theta; U_N) = \prod_{n=1}^N f_w(w_n, \theta; u_n) = \prod_{n=1}^N \int_{\mathcal{Y}} f_z(h_z^{-1}(y, w_n))f_\omega(F_\omega^{-1}(u_n, \theta, y))|J_F||J_h|dy$$



Estimation algorithm: $\theta_N = \Psi_N(U_N, Y_N) \rightarrow L_N(W_N, \theta_N; U_N) = \max_{\theta \in \Theta} L_N(W_N, \theta; U_N)$



Bayesian method

Assumptions:

Additionally θ is value of random variable $\underline{\theta}$, with probability density function $f_{\theta}(\theta)$

For a given parametr θ and input u_n the sequence $y_n, n = 1, 2, \dots, N$ are values of random variable \underline{y} under condition $\underline{\theta} = \theta$ and input is equal u .

$w_n, n = 1, 2, \dots, N$ are values of random variable \underline{w} under condition $\underline{\theta} = \theta$

z_n is value of random variable \underline{z} , with probability density function $f_z(z)$

Calculations:

$$\underline{w} = h(\underline{y}, \underline{z})$$

$$f_w(w|\theta; u) = \int_{\mathcal{Y}} f_z(h_z^{-1}(y, w)) f_y(y|\theta; u) |J| dy = \int_{\mathcal{Y}} f_z(h_z^{-1}(y, w)) f_{\omega}(F_{\omega}^{-1}(u, \theta, y)) |J_F| |J_h| dy$$



Bayesian method

A'posteriori probability density function:

$$f'(\theta|W_N; U_N) = \frac{\int_{\mathcal{Y}} f_{\theta}(\theta) \prod_{n=1}^N \int_{\mathcal{Y}} f_z(h_z^{-1}(y, w_n)) f_{\omega}(F_{\omega}^{-1}(u_n, \theta, y)) |J_F| |J_h| dy}{\int_{\Theta} \int_{\mathcal{Y}} f_{\theta}(\theta) \prod_{n=1}^N \int_{\mathcal{Y}} f_z(h_z^{-1}(y, w_n)) f_{\omega}(F_{\omega}^{-1}(u_n, \theta, y)) |J_F| |J_h| dy d\theta}$$

Conditional risk :

$$r(\bar{\theta}, Y_N; U_N) \stackrel{\text{df}}{=} E_{\bar{\theta}}[L(\theta, \bar{\theta}) | W_N; U_N] = \int_{\Theta} L(\theta, \bar{\theta} = \bar{\Psi}(U_N, W_N)) f'(\theta | W_N; U_N) d\theta$$

Estimation algorithm: $\theta_N = \Psi_N(U_N, W_N) \rightarrow r(\theta_N, W_N; U_N) = \min_{\theta \in \Theta} r(\bar{\theta}, W_N; U_N)$



Thank you for attention

